

Orbits homoclinic to resonances: the Hamiltonian case

György Haller and Stephen Wiggins

Applied Mechanics 104-44, Caltech, Pasadena, CA 91125, USA

Received 27 May 1992

Revised manuscript received 4 November 1992

Accepted 27 December 1992

Communicated by H. Flaschka

In this paper we develop methods to show the existence of orbits homoclinic or heteroclinic to periodic orbits, hyperbolic fixed points or combinations of hyperbolic fixed points and/or periodic orbits in a class of two-degree-of-freedom, integrable Hamiltonian systems subject to arbitrary Hamiltonian perturbations. Our methods differ from previous methods in that the invariant sets (periodic orbits, fixed points) are created, and become hyperbolic, as a result of the interaction of the perturbation with a *resonance* in the unperturbed system. This results in a very degenerate situation that requires a combination of geometric singular perturbation theory, higher-dimensional Melnikov-type methods, and transversality theory. We establish a simple *energy-phase criterion* which gives a fairly complete picture of the complex dynamics associated with orbits homoclinic to the resonance. We apply our methods to a two-mode truncation of the driven nonlinear Schrödinger equation first studied by Bishop et al. In this example we show that as the energy is increased at resonance, orbits homoclinic to hyperbolic periodic orbits are created in pairs in a global bifurcation that is best described as a *saddle-node bifurcation of homoclinic orbits*.

1. Introduction

The development of perturbation methods for finite degree-of-freedom, integrable Hamiltonian systems is an old subject, going back well over 100 years. As with any perturbation method, the structure or form of the unperturbed system plays an important role in the development of the method. Since the unperturbed system is assumed integrable, action-angle variables are particularly useful for expressing it in the most simple form. For an n degree-of-freedom system, this coordinate representation renders transparent the geometrical interpretation that the $2n$ -dimensional phase space is foliated by n -dimensional tori.

Two major theorems resulting from the evolution of perturbation methods for finite degree-of-freedom, integrable Hamiltonian systems in this setting are the Kolmogorov–Arnold–Moser (KAM) theorem and Nekhoroshev’s theorem (see e.g. Arnold et al. [1] for a recent summary and survey of the latest results). The KAM theorem is concerned with the preservation of n -dimensional elliptic tori on which the motion is nonresonant (thus, an *infinite time* result) and Nekhoroshev’s theorem addresses the issue of the deviation of the action variables of the perturbed system from those of the unperturbed system over finite time scales. Recent extensions of “KAM like” results have been obtained by Pöschel [2] and Eliasson [3] who consider the preservation of nonresonant elliptic tori of dimension smaller than n and Treshchev [4] and de la Llave and Wayne [5] who consider the preservation of tori of dimension less than n that are hyperbolic in stability type (i.e. “whiskered tori”). The hyperbolicity in the results of Treshchev and de la Llave and Wayne is created by the perturbation. We note that in all of these

results the unperturbed systems are (essentially) expressed in action-angle variables. This has the important consequence that all dynamical phenomena in the unperturbed system are elliptic in stability type. We will discuss the significance of this statement to our work shortly.

Perturbation results have also been obtained in the situation where the unperturbed system is expressed as a combination of action-angle variables and variables that allow for exponential growth and decay. Graff [6] has considered a “KAM-type” situation where the unperturbed problem has a submanifold foliated by invariant tori. Restricted to this submanifold, the dynamics is elliptic and described by action-angle variables. Transverse to this submanifold, the dynamics is hyperbolic. He proves that a Cantor set of tori is preserved and that the preserved tori have stable and unstable manifolds. Arnold [7] and Holmes and Marsden [8] consider a situation similar to that of Graff, yet with the added condition that the stable and unstable manifolds of the tori intersect nontransversely. They then develop a perturbation method in the spirit of Melnikov [9] for determining the existence of transversal intersection of the stable and unstable manifolds of the non-resonant tori that survive under perturbation. Arnold used these intersecting manifolds in a specific example for the construction of *transition chains* which are the mechanism for what is now referred to as *Arnold diffusion*. Holmes and Marsden’s work removed certain restrictions from Arnold’s work and extended it to a larger class of systems.

In this paper we develop techniques for studying general Hamiltonian perturbations of a class of two-degree-of-freedom, integrable Hamiltonian systems under hypotheses that are quite different from those previously studied. As in the work of Graff, Arnold, and Holmes and Marsden mentioned above, our unperturbed system is expressed in a mixture of action-angle variables and variables that can describe homoclinic and heteroclinic behavior. The key new feature of our analysis is that in the unperturbed system we have a *resonance* in the action-angle variables. We describe our situation more fully.

The geometrical manifestation of our coordinate representation is that the unperturbed integrable system contains a two-dimensional, normally hyperbolic invariant two-manifold that has three-dimensional stable and unstable manifolds that coincide along a branch. The dynamics on the normally hyperbolic two-manifold is described by action-angle variables. Hence, typically, we expect this manifold to be filled with periodic orbits, unless the frequency vanishes at some particular value of the action. We refer to this situation as *resonance* and it leads to a circle of fixed points on the two-manifold. This resonant behavior in the action-angle variables is the focal point of our analysis. Each point on the circle of fixed points is connected (in general) to another fixed point on the circle by a heteroclinic orbit. These heteroclinic orbits are part of a foliation of the three-dimensional stable and unstable manifolds of the normally hyperbolic two-manifold. Evidently, the dynamics near this circle of fixed points can be dramatically altered by the perturbation. We will see that, under general conditions, the perturbation “blows up” the circle of fixed points so that the dynamics restricted to the normally hyperbolic two-manifold near the resonance consists of the standard situation of a finite number of elliptic and hyperbolic fixed points with periodic orbits surrounding the elliptic points and homoclinic or heteroclinic trajectories connecting the hyperbolic points. Our goal will be to give conditions for the existence of orbits in the full four-dimensional phase space that are homoclinic or heteroclinic to the orbits created in the resonance.

There are three separate steps in the process. One involves analyzing the perturbed dynamics, restricted to the normally hyperbolic, invariant two-manifold, near the resonance. The approach here is standard and involves well known “resonance rescalings” from nonlinear oscillation theory. Another step requires showing the existence of orbits homoclinic to the *normally hyperbolic invariant two-*

manifold. Basically, this is accomplished through a higher-dimensional Melnikov-type analysis. The final step is the most difficult and represents the most innovative part of this work. The goal is to “match” orbits homoclinic to the normally hyperbolic invariant two-manifold with trajectories on this manifold in both forward and backward time. This will give rise to the existence of orbits homoclinic or heteroclinic to specific orbits on the two manifold. The analysis is complicated by the presence of a “boundary layer” near the resonance and by the fact that we are concerned with perturbations of nontransversal intersections of manifolds. We analyze this situation using the geometric singular perturbation theory of Fenichel, that is based on foliations of stable and unstable manifolds, in combination with energy-type arguments that are suited for Hamiltonian systems. An interesting result of our analysis is that the existence of these homoclinic and heteroclinic orbits in the full four-dimensional phase space can be shown to depend only on an *energy-phase* criterion that is obtained from a *reduced, one-degree-of-freedom Hamiltonian system*.

The significance of the class of systems under consideration is that they frequently appear in applications. In particular, they arise in studies of the model interactions of parametrically excited plates, shells, and surface waves (see, e.g., Feng and Sethna [10–12], Feng and Wiggins [13], and Yang and Sethna [14]). The techniques developed in this paper are related to the non-Hamiltonian methods for a similar class of systems developed in Kováčić and Wiggins [15]. The motivation for that work was to understand modal interactions in the damped, driven nonlinear Schrödinger equation. We will use the Hamiltonian version of that example to illustrate our results.

This paper is organized as follows. In section 2 we formulate the problem and discuss the *standard form* of the systems under consideration. In section 3 we describe the resulting geometrical structure of the phase space of the unperturbed standard form. In section 4 we discuss some general properties of the perturbed system based on perturbation theory for normally hyperbolic invariant manifolds as well as the *fiber*ing of the stable and unstable manifolds of normally hyperbolic invariant manifolds by one-dimensional curves. These curves consist of initial conditions that approach the same trajectory on the normally hyperbolic invariant manifold. Section 5 contains the main analytical results. In this section we develop the techniques for proving the existence of specific orbits homoclinic or heteroclinic to trajectories near the resonance. In section 6 we discuss mechanisms for chaos associated with these homoclinic orbits and in section 7 we apply our methods to a specific example: a two-mode truncation of the driven nonlinear Schrödinger equation that was derived by Bishop et al. [16,17]. An interesting feature of this example is that as the energy is increased at resonance, orbits homoclinic to hyperbolic periodic orbits are created in pairs in a global bifurcation that is best described as a *saddle-node bifurcation of homoclinic orbits*.

2. The formulation of the problem and the standard form

Let us consider a two-degree-of-freedom Hamiltonian system given by

$$\dot{x} = JD_x H_0(x, I; \mu) + \varepsilon JD_x H_1(x, I, \phi; \mu, \varepsilon), \quad (2.1a)_\varepsilon$$

$$\dot{I} = -\varepsilon D_\phi H_1(x, I, \phi; \mu, \varepsilon), \quad \dot{\phi} = D_I H_0(x, I; \mu) + \varepsilon D_I H_1(x, I, \phi; \mu, \varepsilon), \quad (2.1b)_\varepsilon$$

$$0 \leq \varepsilon \ll 1, \quad (x, I, \phi; \mu) \in \mathcal{P} = \mathbb{R}^2 \times U \times S^1 \times V,$$

where $U \subset \mathbb{R}^+$ and $V \subset \mathbb{R}^p$ are open sets, μ is a p -dimensional vector of parameters, and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

System (2.1) is derived from the Hamiltonian $H = H_0 + \varepsilon H_1$, which is assumed to be C^{r+1} smooth ($r \geq 2$) in its arguments. For $\varepsilon = 0$ system (2.1)₀ will be called the *unperturbed system*, while for $\varepsilon > 0$ (2.1) _{ε} will be referred to as the *perturbed system*. Note that (2.1)₀ is an integrable Hamiltonian system on which we make two structural assumptions:

- (H1) There exist $I_1, I_2 \in U$, $I_1 < I_2$ such that for any $(I, \mu) \in [I_1, I_2] \times V$ (2.1a)₀ has a hyperbolic fixed point $\bar{x}_0(I; \mu)$ and a homoclinic trajectory $x^h(t, I; \mu)$, which connects \bar{x}_0 to itself.
- (H2) (Resonance) There exists $I_r \in (I_1, I_2)$ such that

$$D_I H_0(\bar{x}_0(I_r; \mu), I_r; \mu) = 0, \quad m(I_r; \mu) = D_I^2[H_0(\bar{x}_0(I; \mu), I; \mu)]|_{I=I_r} \neq 0.$$

Remark 2.1. If $m(I_r; \mu)$ happens to be zero, our theory can still be applied provided there exists $n \geq 2$ such that $D_I^i[H_0(\bar{x}_0(I; \mu), I; \mu)]|_{I=I_r} = 0$ if $1 \leq i \leq n-1$, but $D_I^n[H_0(\bar{x}_0(I; \mu), I; \mu)]|_{I=I_r} \neq 0$ (see also remark 2.3).

Remark 2.2. Assumption (H2) can be replaced by

- (H2') (Resonance) For every $I \in [I_1, I_2]$

$$D_I H_0(\bar{x}_0(I; \mu), I; \mu) = 0.$$

A system satisfying assumption (H2') arises e.g. in the study of the Hamiltonian 1:2:2 resonance (see Haller and Wiggins [18]). In fact, even if (H2) holds we will transform (2.1)₀ in a neighborhood of (x, I_r, ϕ) to a form which satisfies (H2') and will work with that form. We will comment on the necessary modifications corresponding to (H2') as we proceed further.

Let us assume now that (H2) holds. We will primarily be interested in the behavior of orbits with their I coordinates in a neighborhood of the value I_r , which we call *resonant*, since the frequency corresponding to the angular variable ϕ vanishes for this value of I at $x = \bar{x}_0(I_r; \mu)$. We can Taylor-expand the Hamiltonian H about I_r to obtain

$$\begin{aligned} H(x, I, \phi; \mu, \varepsilon) &= H_0(x, I_r; \mu) + D_I H_0(x, I_r; \mu) \Delta I + \frac{1}{2} D_I^2 H_0(x, I_r; \mu) (\Delta I)^2 \\ &\quad + \varepsilon H_p(x, I_r, \phi; \mu) + \varepsilon D_I H_p(x, I_r, \phi; \mu) \Delta I + \mathcal{O}((\Delta I)^3, \varepsilon(\Delta I)^2, \varepsilon^2), \end{aligned} \quad (2.2)$$

where we have introduced the notations

$$I = I_r + \Delta I, \quad H_p(x, I_r, \phi; \mu) = H_1(x, I_r, \phi; \mu, 0).$$

We can derive the Hamiltonian equations associated with (2.2) in the canonical way to obtain

$$\begin{aligned} \dot{x} &= J D_x H_0(x, I_r; \mu) + J D_x D_I H_0(x, I_r; \mu) \Delta I + \mathcal{O}(\varepsilon, (\Delta I)^2), \\ \dot{I} &= -\varepsilon D_\phi H_p(x, I_r, \phi; \mu) + \mathcal{O}(\varepsilon \Delta I, \varepsilon^2), \\ \dot{\phi} &= D_I H_0(x, I_r; \mu) + D_I^2 H_0(x, I_r; \mu) \Delta I + \varepsilon D_I H_p(x, I_r, \phi; \mu) + \mathcal{O}((\Delta I)^2, \varepsilon \Delta I, \varepsilon^2). \end{aligned} \quad (2.3)$$

What we have now is a Hamiltonian system localized near the resonant “slice” (x, I_r, ϕ) of the phase space of $(2.1)_\epsilon$. It is valid for trajectories which stay close to this slice but we have not specified yet what we consider close. We now let

$$\Delta I = \eta\sqrt{\epsilon}, \quad \eta \in [-\eta_0, \eta_0], \quad (2.4)$$

with $\eta_0 > 0$ fixed but as yet undetermined constant.

Remark 2.3. In case of the situation described in remark 2.1, one can introduce a different scaling by letting $\Delta I = \eta\epsilon^{1/n}$. We will not pursue the analysis of that case in this paper but adapting all what follows to it would be straightforward.

If we set $\epsilon = 0$ in (2.3) then use (2.4), we obtain the equations

$$\dot{x} = JD_x H_0(x, I_r; \mu), \quad (2.5a)_0$$

$$\dot{I} = 0, \quad \dot{\phi} = D_I H_0(x, I_r; \mu). \quad (2.5b)_0$$

Normally, this system would not give us extra information since by setting $\epsilon = 0$ we restrict the domain of I to the single value I_r (see (2.4)). However, we will *let I vary in equation (2.5)₀ in the interval $[I_r - \sqrt{\epsilon}\eta_0, I_r + \sqrt{\epsilon}\eta_0]$* . As a result of this, we can no longer hope that (2.5)₀ is equivalent to (2.1)₀ in a neighborhood of the resonance and it is not true indeed, as we will see. However, system (2.3) rewritten in the form

$$\begin{aligned} \dot{x} &= JD_x H_0(x, I_r; \mu) + \sqrt{\epsilon}JD_x D_I H_0(x, I_r; \mu)\eta + \mathcal{O}(\epsilon), \\ \dot{I} &= -\epsilon D_\phi H_p(x, I_r, \phi; \mu) + \mathcal{O}(\epsilon^{3/2}), \\ \dot{\phi} &= D_I H_0(x, I_r; \mu) + \sqrt{\epsilon}D_I^2 H_0(x, I_r; \mu)\eta + \mathcal{O}(\epsilon), \\ 0 < \epsilon &\ll 1, \quad (x, I, \phi; \mu) \in \tilde{\mathcal{P}} = \mathbb{R}^2 \times [\tilde{I}_1, \tilde{I}_2] \times S^1 \times V, \end{aligned} \quad (2.5)_\epsilon$$

with $\tilde{I}_1 = I_r - \sqrt{\epsilon}\eta_0$, $\tilde{I}_2 = I_r + \sqrt{\epsilon}\eta_0$ is equivalent to $(2.1)_\epsilon$ near I_r and it can be considered as a perturbation of $(2.5)_0$ (η is a function of I). Hence, we have associated a fictitious unperturbed system of the form $(2.5)_0$ to our original unperturbed problem $(2.1)_0$ in a way, that the associated perturbed system $(2.5)_\epsilon$ coincides with our original perturbed problem $(2.1)_\epsilon$ in a neighborhood of the hyper-surface (x, I_r, ϕ) . We will analyse this associated (singular) perturbation problem which will prove to reveal more features of the behavior of $(2.1)_\epsilon$ near the resonance. Throughout the analysis we will refer to $(2.5)_\epsilon$ as the *standard form*. It can be derived (using our original canonical symplectic form $\omega = dx_1 \wedge dx_2 + d\phi \wedge dI$) from the Hamiltonian H , which we now rewrite from (2.2) as

$$\begin{aligned} H(x, I(\eta, \epsilon), \phi; \mu, \epsilon) &= H_0(x, I_r; \mu) + \sqrt{\epsilon}D_I H_0(x, I_r; \mu)\eta \\ &\quad + \epsilon[\tfrac{1}{2}D_I^2 H_0(x, I_r; \mu)\eta^2 + H_p(x, I_r, \phi; \mu)] + \mathcal{O}(\epsilon^{3/2}). \end{aligned} \quad (2.6)$$

When we derive the Hamiltonian vectorfield corresponding to (2.6), we have to keep in mind, that η is a notation for an expression containing I . At this point, the only reason why we use $\sqrt{\epsilon}\eta$ instead of ΔI is to emphasize its order of magnitude.

We finally point out an important feature of the standard form: it satisfies hypotheses (H1) and (H2') with $I_1 = \tilde{I}_1$, $I_2 = \tilde{I}_2$. In this sense it is similar to $(2.1)_0$ with hypothesis (H2') assumed, the only difference being that the right-hand side of $(2.5)_0$ has no I dependence. As we will see, this similarity will enable us to apply our results (with slight modifications) to the original perturbation problem $(2.1)_\varepsilon$ in the case when assumption (H2') holds.

3. The unperturbed standard form

In the following we would like to discuss some features of $(2.5)_0$ under assumptions (H1) and (H2) which are essential from the point of view of our analysis. We loosely follow the exposition of Kovačič and Wiggins [15] for a similar set-up, which the reader might want to consult for more details.

3.1. Invariant manifolds

Let us fix some small $\varepsilon > 0$. Then, according to (H1), for every $\mu \in V$ system $(2.5)_0$ possesses a two-dimensional invariant manifold (with boundary) \mathcal{A}_0 , defined by

$$\mathcal{A}_0 = \{(x, I, \phi) | x = \bar{x}_0(I_r; \mu), I \in [\tilde{I}_1, \tilde{I}_2], \phi \in S^1\}.$$

\mathcal{A}_0 can be considered as the embedding of the annulus

$$\tilde{A} = [\tilde{I}_1, \tilde{I}_2] \times S^1$$

in the four-dimensional phase space $\tilde{\mathcal{P}}$, a fact which we will use frequently. As a consequence of assumption (H2), all of \mathcal{A}_0 consists of equilibria. We note that \mathcal{A}_0 depends on μ , but this will be suppressed in our notation. It is easy to see that \mathcal{A}_0 has a three-dimensional stable manifold $W^s(\mathcal{A}_0)$ and a three-dimensional unstable manifold $W^u(\mathcal{A}_0)$, which coincide in the homoclinic manifold Γ_0 (see fig.

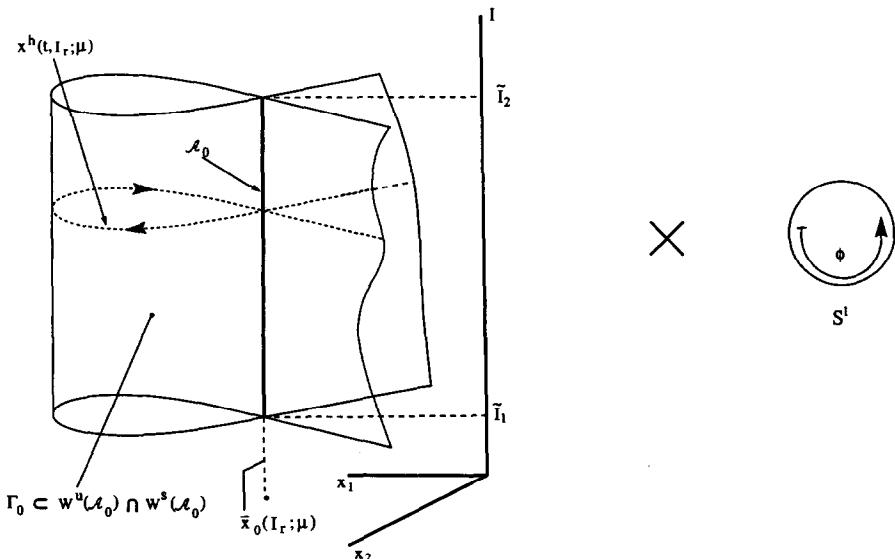


Fig. 1. The geometry of the unperturbed standard form.

1). This degenerate situation (nontransversal intersection of $W^u(\mathcal{A}_0)$ and $W^s(\mathcal{A}_0)$) is the consequence of the integrability of $(2.5)_0$ and changes under generic perturbations in $(2.5)_\varepsilon$.

Consider now a fixed solution $x^h(t, I_r; \mu)$ running on the homoclinic loop of $(2.5a)_0$. This solution induces a natural parametrization on Γ_0 given by

$$\Gamma_0 = \{(x, I, \phi_0) \mid x = x^h(-t_0, I_r; \mu), t_0 \in \mathbb{R}, (I, \phi_0) \in \tilde{\mathcal{A}}\}. \quad (3.1)$$

By uniqueness of solutions, every point in Γ_0 can be uniquely characterized by a triple (t_0, I, ϕ) . Note that t_0 measures the time of flight for the solution $x^h(t, I_r; \mu)$ from the point $x^h(-t_0, I_r; \mu)$ to the fixed reference point $x^h(0, I_r; \mu)$. For more discussion on this parametrization see Wiggins [19].

3.2. The dynamics in Γ_0 and \mathcal{A}_0

Let us examine the dynamics on γ_0 a little more closely. Solutions of $(2.5)_0$ on Γ_0 can be written in the form

$$y_0(t, I, \phi_0; \mu) = \begin{pmatrix} x^h(t, I_r; \mu) \\ I \\ \phi_0 + \int_0^t D_I H_0(x^h(t, I_r; \mu), I_r; \mu) dt \end{pmatrix}, \quad (3.2)$$

which shows that trajectories in Γ_0 approach equilibria in \mathcal{A}_0 asymptotically in forward and backward time. This is a consequence of the fact that as $t \rightarrow \pm\infty$, $D_I H_0(x^h(t, I_r; \mu), I_r; \mu)$ tends to zero exponentially fast (see (H2)), and the resulting improper integral in (3.2) converges. In other words,

$$\lim_{t \rightarrow \pm\infty} y_0(t, I, \phi_0; \mu) = y_0(\pm\infty, I, \phi_0; \mu)$$

exist and $y_0(t, I, \phi_0; \mu)$ lies on an orbit asymptotic to fixed points with the same I coordinate in \mathcal{A}_0 . If $y_0(+\infty, I, \phi_0; \mu) = y_0(-\infty, I, \phi_0; \mu)$ then $y_0(t, I, \phi_0; \mu)$ is a homoclinic solution, otherwise it is a heteroclinic trajectory connecting two different points of \mathcal{A}_0 . The way to decide in which category $y_0(t, I, \phi_0; \mu)$ falls is to compute the net change of the angular variable ϕ along its orbit. This change is given by

$$\Delta\phi(\mu) = \int_{-\infty}^{+\infty} D_I H_0(x^h(t, I_r; \mu), I_r; \mu) dt, \quad (3.3)$$

and will be referred to as the *phase shift*. If $\Delta\phi = 2n\pi$, $n \in \mathbb{Z}$, then all points in \mathcal{A}_0 are homoclinic points, while $\Delta\phi \neq 2n\pi$ yields heteroclinic connections between different points of \mathcal{A}_0 (see fig. 2). Once we have fixed I_r (we can have more than one isolated resonant value in our original problem), $\Delta\phi$ is a function of μ only. Therefore, for a given value of μ , either all the points in \mathcal{A}_0 are homoclinic or all of them are heteroclinic. In either case the phase shifts are the same for all the points in \mathcal{A}_0 .

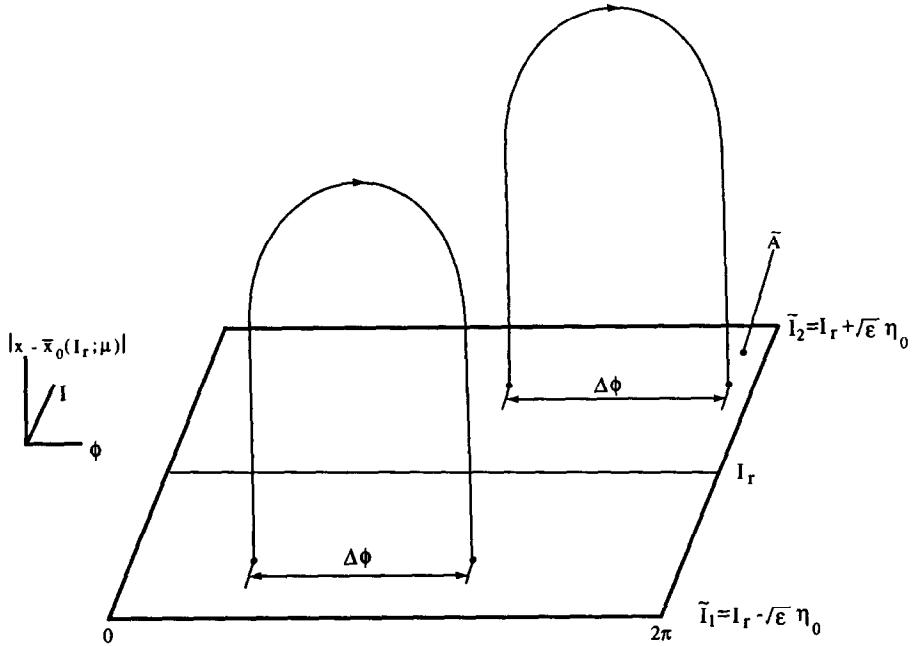


Fig. 2. Heteroclinic orbits with equal phase shifts.

4. The perturbed standard form

4.1. Persistence of the invariant manifolds

In this section we will primarily be concerned with the fate of the invariant manifolds of the integrable structure we have just described under small perturbations of the kind in $(2.5)_\epsilon$. As it is shown in Wiggins [19], manifolds of the type of \mathcal{A}_0 are *normally hyperbolic*. Normal hyperbolicity roughly means that the rate of stretching and contraction in the normal bundle of \mathcal{A}_0 dominates that of the tangent bundle, i.e. the dynamics of $(2.5)_0$ is more “robust” in the directions complementary to \mathcal{A}_0 than on \mathcal{A}_0 . This property can be checked via generalized Liapunov-type numbers defined in Fenichel [20]. What is important for us is that normally hyperbolic invariant manifolds (together with their local stable and unstable manifolds) smoothly persist under small perturbations and are close to the original manifolds. Since we will heavily use these results, we list them more precisely in the form of a proposition. We first define the set

$$U^\delta = \{(x, I, \phi) \mid |x - \bar{x}_0(I_r; \mu)| \leq \delta, (I, \phi) \in \tilde{\mathcal{A}}\},$$

which is a closed tubular neighborhood of \mathcal{A}_0 .

Proposition 4.1. There exists $\epsilon_0 > 0$ such that for every $\epsilon \in [0, \epsilon_0]$ there exists a two-dimensional normally hyperbolic manifold (with boundary) \mathcal{A}_ϵ , which has the following properties:

- (i) \mathcal{A}_ϵ is C' diffeomorphic to \mathcal{A}_0 and can be represented as a graph in the form

$$\mathcal{A}_\varepsilon = \{(x, I, \phi) | x = \bar{x}_\varepsilon(I, \phi; \mu) = \bar{x}_0(I; \mu) + \mathcal{O}(\varepsilon), (I, \phi) \in \tilde{\mathcal{A}}\},$$

where $\bar{x}_\varepsilon(I, \phi; \mu)$ is C' smooth in I, ϕ, ε and μ .

- (ii) \mathcal{A}_ε is locally invariant under the flow of the standard form $(2.5)_\varepsilon$, i.e. solutions starting in \mathcal{A}_ε can leave \mathcal{A}_ε only through $\partial\mathcal{A}_\varepsilon$.

Moreover, there exist three-dimensional manifolds $W_{loc}^s(\mathcal{A}_\varepsilon)$ and $W_{loc}^u(\mathcal{A}_\varepsilon)$, such that

- (iii) $\bar{W}_{loc}^s(\mathcal{A}_\varepsilon) \cap \bar{W}_{loc}^u(\mathcal{A}_\varepsilon) = \mathcal{A}_\varepsilon$
- (iv) There exist $\delta_0 > 0$ such that $W_{loc}^{s,u}(\mathcal{A}_\varepsilon) \cap U^{\delta_0}$ is C' ε -close to $W_{loc}^{s,u}(\mathcal{A}_0) \cap U^{\delta_0}$.
- (v) $W_{loc}^s(\mathcal{A}_\varepsilon)$ and $W_{loc}^u(\mathcal{A}_\varepsilon)$ are locally invariant, i.e. any solution of $(2.5)_\varepsilon$ starting in $W_{loc}^s(\mathcal{A}_\varepsilon) \cap U^{\delta_0}$ (respectively $W_{loc}^u(\mathcal{A}_\varepsilon) \cap U^{\delta_0}$) either approaches \mathcal{A}_ε as $t \rightarrow +\infty$ (respectively $t \rightarrow -\infty$) or crosses ∂U^{δ_0} .

Proof. The proof follows from the persistence theorem of Fenichel [20] applied to $(2.1)_\varepsilon$, as it is explained in detail in Wiggins [19]. See also Kovačič and Wiggins [15] for more discussion on this topic. \square

Remark 4.1. The locally invariant local stable and unstable manifolds of proposition 4.1 can be extended to global (locally invariant) stable and unstable manifolds in the usual way:

$$W^s(\mathcal{A}_\varepsilon) = \cup_{t \leq 0} \phi_t^\varepsilon(W_{loc}^s(\mathcal{A}_\varepsilon) \cap U^{\delta_0}), \quad W^u(\mathcal{A}_\varepsilon) = \cup_{t \geq 0} \phi_t^\varepsilon(W_{loc}^u(\mathcal{A}_\varepsilon) \cap U^{\delta_0}),$$

where $\phi_t^\varepsilon(\cdot)$ denotes the phase flow map of $(2.5)_\varepsilon$.

4.2. The dynamics in $W_{loc}^s(\mathcal{A}_\varepsilon)$ and $W_{loc}^u(\mathcal{A}_\varepsilon)$

We have now some understanding of the fate of the geometric structure of $(2.5)_0$ under perturbation but it is far from complete. We will also need to know more about the behavior of trajectories within $W_{loc}^s(\mathcal{A}_\varepsilon)$ and $W_{loc}^u(\mathcal{A}_\varepsilon)$. At this stage we know that they either leave these manifolds in finite time crossing the “top” or the “bottom” of U^{δ_0} or they are asymptotic to \mathcal{A}_ε , but we have no means to decide what actually happens to given trajectories. A more tight control over the behavior of trajectories can be gained from a result of Fenichel [21] concerning singular perturbation problems of this kind. Fenichel was able to prove the existence and persistence of a smooth family of curves, called *fibers*, which foliate $W_{loc}^{s,u}(\mathcal{A}_\varepsilon)$. The fibers of the family are usually not invariant individually under the flow but the family itself is invariant, i.e. fibers are mapped to fibers by the flow. Each fiber intersects \mathcal{A}_ε in a unique point, which we call the *basepoint of the fiber*, and fibers of the unperturbed problem deform smoothly into fibers of the perturbed problem. The remarkable feature of this fibering is the following: a solution starting on a fiber of $W_{loc}^s(\mathcal{A}_\varepsilon)$ will be asymptotic to the trajectory in \mathcal{A}_ε , which runs through the basepoint of the fiber, as long as the latter one stays in \mathcal{A}_ε (a similar statement holds for $W_{loc}^u(\mathcal{A}_\varepsilon)$). This fact makes it possible to predict the asymptotic behavior of orbits in $W_{loc}^{s,u}(\mathcal{A}_\varepsilon)$, which will be of central importance to us. For this reason, we again formulate the necessary details of this informal discussion more precisely, partly following Kovačič and Wiggins [15]. We will state the results for the fibering of $W_{loc}^s(\mathcal{A}_\varepsilon)$ but, of course, similar statements are true for $W_{loc}^u(\mathcal{A}_\varepsilon)$. *From now on, if we speak about the distance of objects in the phase space, denoted by $d(\cdot, \cdot)$, we will mean it, unless we say otherwise, to be measured in the (x, I, ϕ) coordinates, as opposed to the variables (x, η, ϕ) .*

Proposition 4.2. There exist $\delta_0, \varepsilon_0 > 0$ such that for every $\varepsilon \in [0, \varepsilon_0]$ there exists a two-parameter family $\mathcal{F}_\varepsilon^s = \cup_{p \in \mathcal{A}_\varepsilon} f_\varepsilon^s(p)$ of C' smooth curves (*stable fibers*) $f_\varepsilon^s(p)$ (with boundary), such that the

following hold:

- (i) $\mathcal{F}_\varepsilon^s = (W_{loc}^s(\mathcal{A}_\varepsilon) \cup \mathcal{A}_\varepsilon) \cap U^{\delta_0}$ and $f_\varepsilon^s(p) \cap \mathcal{A}_\varepsilon = p$.
- (ii) $\mathcal{F}_\varepsilon^s$ is C^{r-1} in p (the basepoint of the fiber $f_\varepsilon^s(p)$) and in ε .
- (iii) $\mathcal{F}_\varepsilon^s$ is a positively invariant family, i.e. $\phi_t^\varepsilon(f_\varepsilon^s(p)) \subset f_\varepsilon^s(\phi_t^\varepsilon(p))$ for any $t \geq 0$ and $p \in \mathcal{A}_\varepsilon$ with $\phi_t^\varepsilon(p) \in \mathcal{A}_\varepsilon$.
- (iv) There exist $C_s, \lambda_s > 0$ such that if $y \in f_\varepsilon^s(p)$, then $|\phi_t^\varepsilon(y) - \phi_t^\varepsilon(p)| < C_s e^{-\lambda_s t}$ for any $t \geq 0$ with $\phi_t^\varepsilon(p) \in \mathcal{A}_\varepsilon$.

Proof. See Fenichel [21] but watch for the differences in notation and terminology. See also Kovacić and Wiggins [15] for more features of the fibering and a detailed example. \square

Remark 4.2. It is important to realize that \mathcal{F}_0^s is just the family of (pieces of) orbits of the unperturbed standard form which form $W_{loc}^s(\mathcal{A}_0)$. Hence, the stable fibers of \mathcal{A}_0 are just the unperturbed trajectories in $W_{loc}^s(\mathcal{A}_0)$ and the basepoints of these fibers are the equilibrium points in \mathcal{A}_0 to which these trajectories are asymptotic in forward time (see section 3.2). This enables us to find out more about the orbit structure of $W_{loc}^s(\mathcal{A}_\varepsilon)$. Let $y_0(t, I, \phi_0; \mu)$ be a solution of $(2.5)_0$ (see (3.2)). We assume that t is big enough so that $y_0(t, I, \phi_0; \mu)$ has already entered U^{δ_0} (see proposition 4.2). Thus the orbit containing $y_0(t, I, \phi_0; \mu)$ is exactly the fiber $f_0^s(p_0)$ with basepoint $p_0 = y_0(+\infty, I, \phi_0; \mu)$. Let us consider an orbit y_ε of the perturbed standard form $(2.5)_\varepsilon$ and assume that $y_0(t, I, \phi_0; \mu)$ is ε -close to y_ε outside U^{δ_0} (since $y_0(t, I, \phi_0; \mu)$ spends a finite time outside U^{δ_0} , it is feasible to assume the existence of such a y_ε , as shown in fig. 3). Then, according to (ii) of proposition 4.2, $d(p_0, p_\varepsilon) = \mathcal{O}(\varepsilon)$. Moreover,

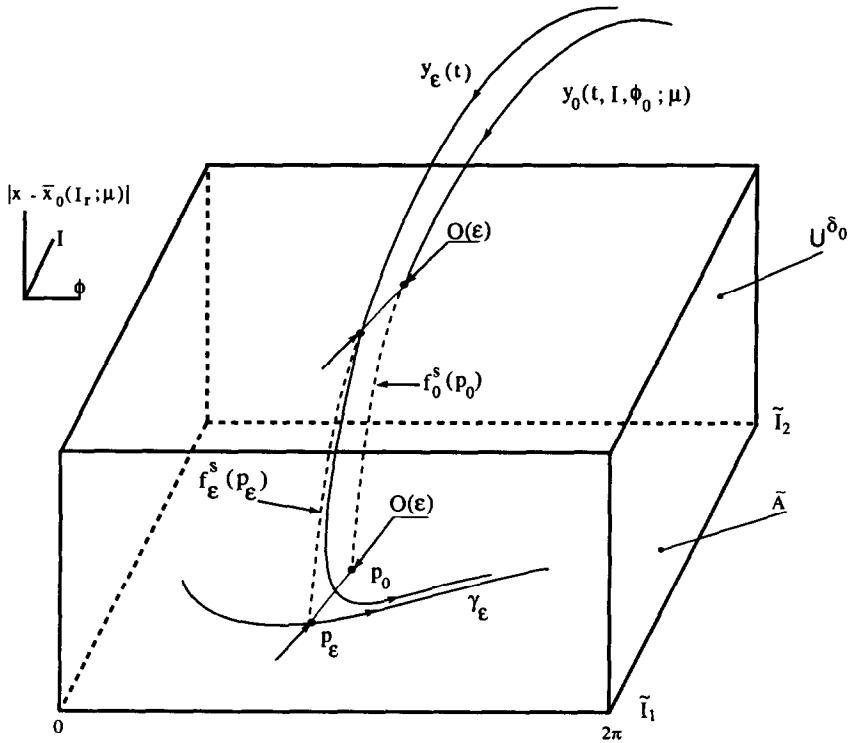


Fig. 3. The significance of fibering.

if $\gamma_\varepsilon \in \mathcal{A}_\varepsilon$ is an orbit of the perturbed standard form with $p_\varepsilon \in \gamma_\varepsilon$ then, by (iv) of proposition 4.2, a solution $y_\varepsilon(t)$ in γ_ε is asymptotic to a solution $\gamma_\varepsilon(t)$ in γ_ε , as t increases, as long as $\gamma_\varepsilon(t)$ stays in \mathcal{A}_ε (see fig. 3). In particular, if the orbits crossing an $\mathcal{O}(\varepsilon)$ neighborhood of p_0 in \mathcal{A}_ε all stay in \mathcal{A}_ε then $y_\varepsilon(t)$ will be asymptotic to \mathcal{A}_ε , too. More specifically, if, e.g., all the orbits crossing a neighborhood of p_0 are periodic orbits then $y_\varepsilon(t)$ will be asymptotic to a periodic orbit of \mathcal{A}_ε for small enough positive ε .

4.3. The dynamics in \mathcal{A}_ε

In order to be able to explore the dynamics near \mathcal{A}_ε via the idea of fibering, we need to know the dynamics on \mathcal{A}_ε itself, as it is clearly seen from remark 4.2. The question we need to answer is whether the dynamics on \mathcal{A}_ε is Hamiltonian and if so, how can we describe it at least approximately. We first prove a technical lemma which will not be stated in its full generality, but rather in a form adapted to our problem. Throughout the lemma d will refer to exterior differentiation and $i_X \omega$ will denote the interior product of the vectorfield X with the form ω .

Lemma 4.3. Let (\mathcal{P}, ω) be a four-dimensional symplectic C^r manifold with (locally defined) coordinates $x = (x_1, x_2) \in \mathbb{R}^2$ and $z = (z_1, z_2) \in \mathbb{R}^2$, and with the symplectic form $\omega = dx_1 \wedge dx_2 + dz_1 \wedge dz_2$. Consider a one parameter family of two-dimensional C^k submanifolds $\mathcal{A}_\varepsilon \subset \mathcal{P}$, with $1 \leq k \leq r$ and $\varepsilon \in [0, \varepsilon_0]$, of the form

$$\mathcal{A}_\varepsilon = \{(x, z) \in \mathcal{P} \mid x = f(z_2) + \varepsilon g(z; \varepsilon)\},$$

where $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are (locally defined) C^{k+1} functions with $f_j, g_j: \mathbb{R}^2 \rightarrow \mathbb{R}$, $j = 1, 2$. We further assume that H is a C^2 function on \mathcal{P} , and for any $\varepsilon \in [0, \varepsilon_0]$, \mathcal{A}_ε is an integral manifold for the Hamiltonian vectorfield $X_H: \mathcal{P} \rightarrow T\mathcal{P}$ defined by

$$i_{X_H} \omega = dH.$$

Then, for ε sufficiently small,

- (i) $(\mathcal{A}_\varepsilon, \bar{\omega}_\varepsilon)$ is a two-dimensional symplectic C^k manifold with $\bar{\omega}_\varepsilon = \omega|_{\mathcal{A}_\varepsilon}$,
- (ii) $X_\varepsilon = X_H|_{\mathcal{A}_\varepsilon}$ is a Hamiltonian vectorfield on \mathcal{A}_ε with Hamiltonian $\mathcal{H}_\varepsilon = H|_{\mathcal{A}_\varepsilon}$, i.e.

$$i_{X_\varepsilon} \bar{\omega}_\varepsilon = d\mathcal{H}_\varepsilon.$$

Proof. We first show that $\bar{\omega}_\varepsilon$ defines a symplectic structure on \mathcal{A}_ε . On \mathcal{A}_ε we have

$$dx_j = \varepsilon D_{z_1} g_j dz_1 + D_{z_2} (f_j + \varepsilon g_j) dz_2, \quad j = 1, 2,$$

implying

$$\begin{aligned} \bar{\omega}_\varepsilon &= \varepsilon (D_{z_1} g_1 D_{z_2} (f_2 + \varepsilon g_2) - D_{z_2} (f_1 + \varepsilon g_1) D_{z_1} g_2) dz_1 \wedge dz_2 + dz_1 \wedge dz_2 \\ &= (1 + \mathcal{O}(\varepsilon)) dz_1 \wedge dz_2, \end{aligned} \tag{4.1}$$

from which we conclude that, for ε sufficiently small, $\bar{\omega}_\varepsilon$ is a nondegenerate 2-form on \mathcal{A}_ε . Let $e_\varepsilon = (f + \varepsilon g, \text{Id}_2): \mathbb{R}^2 \rightarrow \mathcal{P}$ be the embedding of \mathcal{A}_ε . Then

$$d\bar{\omega}_\varepsilon = d(e_\varepsilon^* \omega) = e_\varepsilon^* d\omega = 0, \quad (4.2)$$

which shows that $\bar{\omega}_\varepsilon$ is closed (e_ε^* denotes the pull-back of e_ε). But (4.1) and (4.2) together imply statement (i) of the lemma.

To show (ii) we consider an arbitrary $p_\varepsilon \in e_\varepsilon^{-1}(\mathcal{A}_\varepsilon)$ and $u \in T_p \mathbb{R}^2$. We can write

$$\begin{aligned} i_{X_\varepsilon} \bar{\omega}_\varepsilon [p_\varepsilon](u) &= e^* \omega [p_\varepsilon](e_\varepsilon^* X_H(p_\varepsilon), u) = \omega [e_\varepsilon(p_\varepsilon)](de_\varepsilon de_\varepsilon^{-1} X_H(e_\varepsilon(p_\varepsilon)), de_\varepsilon u) \\ &= \omega [e_\varepsilon(p_\varepsilon)](X_H(e_\varepsilon(p_\varepsilon)), de_\varepsilon u) = dH[e_\varepsilon(p_\varepsilon)](de_\varepsilon u) = d(e_\varepsilon^* H)[p_\varepsilon](u) \\ &= d\mathcal{H}_\varepsilon[p_\varepsilon](u), \end{aligned}$$

which concludes the proof. \square

Remark 4.3. The dimensional restriction on z in lemma 4.3 is not essential but simplifies the proof of (i). Furthermore, f can be a function of z_1 only.

Remark 4.4. Lemma 4.3 is easy but not trivial because it is obviously not true for *arbitrary* invariant manifolds of the Hamiltonian dynamics generated by H , e.g. for stable and unstable manifolds of equilibria. These latter manifolds cannot be viewed as graphs over two variables which are canonically conjugate, as it is readily seen from the structure of the stable and unstable subspaces of the equilibrium in the linearized problem (see also Lerman and Umanskii [22] for a more specific result on the form of the stable and unstable manifolds of equilibria on a four-dimensional symplectic manifold).

We now return to system $(2.5)_\varepsilon$ and use the previous lemma to obtain the following

Proposition 4.4. Let us consider the standard system $(2.5)_\varepsilon$ restricted to its invariant manifold \mathcal{A}_ε . For ε small enough, the restricted dynamics is Hamiltonian with Hamiltonian

$$\mathcal{H}_\varepsilon(I, \phi; \mu)|_{I=I_r + \sqrt{\varepsilon}\eta} = H_0(\bar{x}_0(I_r; \mu), I_r; \mu) + \varepsilon[\frac{1}{2}m(I_r; \mu)\eta^2 + H_p(\bar{x}_0(I_r; \mu), I_r, \phi; \mu)] + \mathcal{O}(\varepsilon^{3/2}), \quad (4.3)$$

with $m(I_r; \mu)$ being the same as in hypothesis (H2) of section 2, and with the restricted symplectic form

$$\bar{\omega}_\varepsilon = (1 + \mathcal{O}(\varepsilon))d\phi \wedge dI. \quad (4.4)$$

Proof. We can directly apply lemma 4.3 with $z_1 = \phi$, $z_2 = I$ and $f = \bar{x}_0(I_r; \mu)$ to obtain (4.4). Furthermore, from proposition 4.1 and lemma 4.3 we see that

$$\begin{aligned} \mathcal{H}_\varepsilon(I, \phi; \mu) &= H(\bar{x}_\varepsilon(I, \phi; \mu), I, \phi; \mu, \varepsilon) \\ &= H_0(\bar{x}_0(I; \mu), I; \mu) + \varepsilon H_p(\bar{x}_0(I; \mu), I, \phi; \mu) + \mathcal{O}(\varepsilon^2), \end{aligned} \quad (4.5)$$

where we have used (H1). If we Taylor-expand (4.5) in I about $I = I_r$, and use proposition 4.1, (H1) and (H2), we obtain (4.3). \square

We now make use of the suggestive notation defined in (2.4) introducing the change of variables

$$I \rightarrow I_r + \sqrt{\varepsilon}\eta, \quad \phi \rightarrow \phi.$$

This change of variables is not canonical since

$$(1 + \mathcal{O}(\varepsilon))d\phi \wedge dI = \sqrt{\varepsilon}(1 + \mathcal{O}(\varepsilon))d\phi \wedge d\eta , \quad (4.6)$$

but this also shows that if we introduce the new restricted Hamiltonian

$$\begin{aligned} \bar{\mathcal{H}}_\varepsilon(\eta, \phi; \mu) &= \frac{1}{\sqrt{\varepsilon}(1 + \mathcal{O}(\varepsilon))} (\mathcal{H}_\varepsilon(I(\eta; \varepsilon), \phi; \mu) - H_0(\bar{x}_0(I_r; \mu), I_r; \mu)) \\ &= \sqrt{\varepsilon}\mathcal{H}(\eta, \phi; \mu) + \mathcal{O}(\varepsilon) , \end{aligned} \quad (4.7)$$

with

$$\mathcal{H}(\eta, \phi; \mu) = \frac{1}{2}m(I_r; \mu)\eta^2 + H_p(\bar{x}_0(I_r; \mu), I_r, \phi; \mu) \quad (4.8)$$

(the $\mathcal{O}(\varepsilon)$ term is just the same that appeared in (4.6)) then we can derive the leading order Hamiltonian dynamics of \mathcal{A}_ε from (4.7) in the canonical way (i.e. using the symplectic form $d\phi \wedge d\eta$) to obtain

$$\dot{\eta} = -\sqrt{\varepsilon}D_\phi \mathcal{H}(\eta, \phi; \mu) + \mathcal{O}(\varepsilon) , \quad \dot{\phi} = \sqrt{\varepsilon}D_\eta \mathcal{H}(\eta, \phi; \mu) + \mathcal{O}(\varepsilon) , \quad (4.9)$$

which we call the *restricted system*. Rescaling the time in (4.9) by letting $\tau = \sqrt{\varepsilon}t$, denoting $d/d\tau$ by ' and setting $\varepsilon = 0$, we arrive at the equations

$$\eta' = -D_\phi \mathcal{H}(\eta, \phi; \mu) , \quad \phi' = D_\eta \mathcal{H}(\eta, \phi; \mu) , \quad (4.10)$$

with Hamiltonian $\mathcal{H}(\eta, \phi; \mu)$, which we call the *reduced system* (see Fenichel [21]). We will consider the reduced system to be defined on the annulus $A = [-\eta_0, \eta_0] \times S^1$. Notice that (4.10) describes a simple one-degree-of-freedom potential problem: $m(I_r; \mu)$ can be thought of as a mass of a particle and $H_p(\bar{x}_0(I_r; \mu), I_r, \phi; \mu)$ can be viewed as the potential of the forces acting on the particle. Measured in the (η, ϕ) coordinates, the orbits of this potential problem perturb by an amount of $\mathcal{O}(\sqrt{\varepsilon})$ into the orbits of (4.9), so just by analyzing this simple system we can obtain some information about the orbit structure of the restricted problem (4.9). However, we have to be careful at this point for two reasons:

1. Orbitas of the reduced system might leave the annulus A in finite time or might perturb to orbits of this kind in (4.9). In either case, the part of the orbit outside A is meaningless for us.
2. Certain orbits of the reduced system (e.g. orbits homoclinic to singular equilibria, heteroclinic orbits) might not perturb smoothly into nearby orbits of (4.9).

To tackle these problems we introduce the following

Definition 4.1. We say that an orbit γ of some Hamiltonian system defined on $A = [-\eta_0, \eta_0] \times S^1$ is an internal orbit if both of the following are satisfied:

- (i) γ is either a periodic orbit or an orbit homoclinic to a hyperbolic fixed point,
- (ii) γ is bounded away from ∂A .

Since for small ε \mathcal{A}_ε is an embedding of \tilde{A} , hence diffeomorphic to A , we will interchangeably speak about internal orbits in A , \tilde{A} and \mathcal{A}_ε .

Remark 4.5. γ is considered to be a homoclinic orbit if it connects points which have equal η

coordinates and equal ϕ coordinates mod 2π . In specific applications one can also allow γ to be a heteroclinic orbit if it is structurally stable with respect to the class of perturbations considered.

It is clear from definition 4.1 that internal orbits do not intersect nor are they asymptotic to the boundary of A , and this property is obviously preserved for sufficiently small perturbations. Furthermore, in case of small perturbation of the Hamiltonian system, internal orbits deform smoothly into nearby orbits of the perturbed system. The reason why we have also excluded nonsingular equilibria from the definition of internal orbits will become apparent shortly.

Finally, we remark that the restricted and reduced systems can also be derived in the case when assumption (H2') holds, but the reduced system will not be a potential problem in general. We will see this in more detail in section 5.4.

5. The existence of orbits homoclinic to the resonance

In this section we will give conditions for the existence of orbits of the standard form $(2.5)_\epsilon$ which are asymptotic to internal orbits of the restricted system (4.9) in forward and backward time. Our goal will be to formulate these conditions in terms of the reduced system, hence simplify our original problem to the analysis of a simple potential problem in the neighborhood of the resonance. To achieve this, we have to discuss several geometric and technical issues. We will primarily be concerned with the transversal intersection of certain invariant manifolds of the standard form and frequently refer to the following trivial but important observation:

Remark 5.1. Let us suppose that \mathcal{S}_1 and \mathcal{S}_2 are two manifolds invariant under the flow of the standard form. Let us further assume that $p \in \mathcal{S}_1 \cap \mathcal{S}_2$, and \mathcal{S}_1 and \mathcal{S}_2 intersect transversally at p . Then \mathcal{S}_1 and \mathcal{S}_2 intersect transversally all along the orbit of the standard form which contains p . This readily follows from the invariance of the manifolds, and from the fact that the phase flow map is a diffeomorphism. (The same statement may not be true for the closure of the orbit through p).

Before starting the analysis of this chapter, we point out one more time that for small $\epsilon \geq 0$ \mathcal{A}_ϵ is a C' embedding of the annulus A in the four-dimensional phase space $\tilde{\mathcal{P}}$. We will sometimes refer to points or orbits in \mathcal{A}_ϵ and then consider them in another context as points or orbits in A . This should cause no confusion by the one-to-one correspondence between the two manifolds for fixed ϵ . We will consider different dynamics on \mathcal{A}_ϵ : the flow of the reduced system and the flow of the restricted system. From the viewpoint of perturbation theory, it is quite natural to “copy” these dynamics together in the annulus A and examine how close certain orbits of the restricted system are to certain orbits of the reduced system.

5.1. The intersection of \mathcal{A}_ϵ , $W^s(\mathcal{A}_\epsilon)$ and $W^u(\mathcal{A}_\epsilon)$ with $E_\epsilon(h)$

Let us start by defining

$$E_\epsilon(h) = \{(x, I, \phi) \mid H(x, I, \phi; \mu, \epsilon) = h\}$$

to be the *perturbed energy surface* with energy h (see (2.6) for the form of H we use). In the following we will examine the intersection of \mathcal{A}_ϵ with certain perturbed energy surfaces.

First, we need an easy technical lemma, which will be stated in a form incorporating the case of assumption (H2'). Namely, we will allow the graph of \mathcal{A}_0 (i.e. \bar{x}_0) to depend explicitly on I as well. *From this point, D (without subscript) will refer to the gradient operator in the variables (x, I, ϕ) .*

Lemma 5.1.

$$DH|_{\mathcal{A}_\varepsilon} = (-\varepsilon D_\phi \mathcal{H}(D_I \bar{x}_0)^T J, \sqrt{\varepsilon} D_\eta \mathcal{H}, \varepsilon D_\phi \mathcal{H}) + \mathcal{O}(\varepsilon^{3/2}).$$

Proof. Let us consider first the x -component of $DH|_{\mathcal{A}_\varepsilon}$. Differentiating \bar{x}_ε (see proposition 4.1) with respect to t we have

$$\begin{aligned} JD_x H|_{\mathcal{A}_\varepsilon} &= \dot{x}|_{\mathcal{A}_\varepsilon} = (D_I \bar{x}_0 + \mathcal{O}(\varepsilon))(-D_\phi \mathcal{H}_\varepsilon) \\ &= -\varepsilon D_\phi \mathcal{H} D_I \bar{x}_0 + \mathcal{O}(\varepsilon^{3/2}), \end{aligned} \quad (5.1)$$

where we used (4.3) and (4.7). If we left-multiply both sides of (5.1) by $J^{-1} = -J$, we obtain

$$D_x H|_{\mathcal{A}_\varepsilon} = \varepsilon D_\phi \mathcal{H} J D_I \bar{x}_0 + \mathcal{O}(\varepsilon^{3/2}),$$

which, after transposition, proves the statement of the lemma for the first two components of $DH|_{\mathcal{A}_\varepsilon}$. For the second two components, using the notation of lemma 4.3, we can write

$$D_{(I,\phi)} H|_{\mathcal{A}_\varepsilon} = e^*(dH) = d(e^* H) = d\mathcal{H}_\varepsilon = (\sqrt{\varepsilon} D_\eta \mathcal{H}, \varepsilon D_\phi \mathcal{H}) + \mathcal{O}(\varepsilon^{3/2}),$$

which completes the proof. \square

Now we turn to the question of the intersection of $E_\varepsilon(h)$ and \mathcal{A}_ε and formulate the following

Proposition 5.2. Let us assume that γ_0 is an internal orbit of the reduced system (4.10) with $\mathcal{H}|_{\gamma_0} = h_0$. Then, for any $p \in \gamma_0$ there exists $\varepsilon_0 > 0$ and an open neighborhood $U_p \subset A$ with $p \in U_p$ such that for $0 < \varepsilon < \varepsilon_0$ the following are satisfied:

- (i) Any orbit $\gamma_\varepsilon \subset A$ of the restricted system (4.9) with $\gamma_\varepsilon \cap U_p \neq \emptyset$ is an internal orbit. Moreover, if $\mathcal{H}_\varepsilon|_{\gamma_\varepsilon} = h$ then $E_\varepsilon(h)$ intersects \mathcal{A}_ε transversally in γ_ε .
- (ii) There exists an orbit $\gamma_\varepsilon^* \subset A$ with $\gamma_\varepsilon^* \cap U_p \neq \emptyset$ which is $C^{r-1} \sqrt{\varepsilon}$ -close to γ_0 .

Proof: By definition, internal orbits are structurally stable which, together with (4.9), immediately implies (ii) of the proposition. Also, since internal orbits are not isolated, statement (i) is obvious with the exception of transversality.

Let us consider an orbit γ_ε with $\gamma_\varepsilon \cap U_p \neq \emptyset$ and $\mathcal{H}_\varepsilon|_{\gamma_\varepsilon} = h$. We want to show that the energy surface $E_\varepsilon(h)$ intersects \mathcal{A}_ε transversally in γ_ε . The manifold \mathcal{A}_ε satisfies the equation

$$x = \bar{x}_\varepsilon(I, \phi; \mu), \quad (5.2a)$$

while the energy surface $E_\varepsilon(h)$ is described by

$$H(x, I, \phi; \mu, \varepsilon) = h. \quad (5.2b)$$

For the transversal intersection of the two manifolds it suffices to show that at the intersection the

defining functions in (5.2) are independent. Therefore, it is sufficient for us to show that, for small ε , $0 \in \mathbb{R}^3$ is a regular value for the map

$$G_\varepsilon: U_\varepsilon \rightarrow \mathbb{R}^3, \quad \begin{pmatrix} x \\ I \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} x - \bar{x}_\varepsilon(I, \phi; \mu) \\ H(x, I, \phi; \mu, \varepsilon) - h \end{pmatrix},$$

where U_ε is an open set of $\tilde{\mathcal{P}}$ which does not contain points of $\mathcal{A}_\varepsilon \cap E_\varepsilon(h)$ other than those contained in γ_ε . It is easy to see that

$$DG_\varepsilon = \begin{pmatrix} \text{Id}_2 - D_{(I, \phi)} \bar{x}_\varepsilon \\ DH \end{pmatrix}, \quad (5.3)$$

where Id_2 denotes the two-dimensional identity matrix. We now select a $p_\varepsilon \in \gamma_\varepsilon$. Then, for ε small enough there exists $p_0 \in U_p$ such that p_0 lies in an internal orbit $\tilde{\gamma}_0$ of the reduced system, which is C^{r-2} -close to γ_ε . We want to show that $DG_\varepsilon|_{p_\varepsilon}$ is surjective. Since γ_ε is C^{r-2} -close to $\tilde{\gamma}_0$, we can Taylor-expand (5.3) at p_0 which, together with lemma 5.1, yields

$$DG_\varepsilon|_{p_\varepsilon} = \begin{pmatrix} 1 & 0 & -D_I \bar{x}_0^1 + \mathcal{O}(\sqrt{\varepsilon}) & \mathcal{O}(\sqrt{\varepsilon}) \\ 0 & 1 & -D_I \bar{x}_0^2 + \mathcal{O}(\sqrt{\varepsilon}) & \mathcal{O}(\sqrt{\varepsilon}) \\ \varepsilon D_\phi \mathcal{H} D_I \bar{x}_0^2 + \mathcal{O}(\varepsilon^{3/2}) & -\varepsilon D_\phi \mathcal{H} D_I \bar{x}_0^1 + \mathcal{O}(\varepsilon^{3/2}) & \sqrt{\varepsilon} D_\eta \mathcal{H} + \mathcal{O}(\varepsilon^{3/2}) & \varepsilon D_\phi \mathcal{H} + \mathcal{O}(\varepsilon^{3/2}) \end{pmatrix}, \quad (5.4)$$

where the leading order terms of the right-hand side are evaluated at $p_0 \in \tilde{\gamma}_0$, and we have used the notation $\bar{x}_0 = (\bar{x}_0^1, \bar{x}_0^2)$.

It is easy to check that the determinant of the minor which contains the first three columns of (5.4) equals $\sqrt{\varepsilon} D_\eta \mathcal{H}|_{p_0} + \mathcal{O}(\varepsilon^{3/2})$, while the determinant computed on the first, second and fourth columns of (5.4) equals $\varepsilon D_\phi \mathcal{H}|_{p_0} + \mathcal{O}(\varepsilon^{3/2})$. Since p_0 is not an equilibrium (this is why equilibria were not considered in definition 4.1), one of these determinants is always nonzero, hence $DG_\varepsilon|_{p_\varepsilon}$ is surjective for small positive ε . But then remark 5.1 completes the proof. \square

Using proposition 5.2 we can obtain information about the intersection of certain energy surfaces with $W^s(\mathcal{A}_\varepsilon)$ and $W^u(\mathcal{A}_\varepsilon)$. We can actually prove the following

Proposition 5.3. Let us assume that γ_0 is an internal orbit of the reduced system (4.10). Then, for any $p \in \gamma_0$ there exists $\varepsilon_0 > 0$ and an open neighborhood $U_p \subset A$ with $p \in U_p$ such that for $0 < \varepsilon < \varepsilon_0$ the following are satisfied:

- (i) For any γ_ε orbit of the restricted system with $\gamma_\varepsilon \cap U_p \neq \emptyset$ and $H_\varepsilon|_{\gamma_\varepsilon} = h$, the energy surface $E_\varepsilon(h)$ intersects $W^s(\mathcal{A}_\varepsilon)$ and $W^u(\mathcal{A}_\varepsilon)$.
- (ii) At least one connected component of the intersection $W^s(\mathcal{A}_\varepsilon) \cap E_\varepsilon(h)$ (respectively $W^u(\mathcal{A}_\varepsilon) \cap E_\varepsilon(h)$) is transversal and identical to $W^s(\gamma_\varepsilon)$ (respectively $W^u(\gamma_\varepsilon)$).
- (iii) For sufficiently small δ_0 with $0 < \varepsilon \ll \delta_0 \ll 1$, $W^s(\gamma_\varepsilon) \cup W^u(\gamma_\varepsilon) \cap U^{\delta_0}$ is approximated with an error of $\mathcal{O}(\varepsilon, \delta_0^2)$ by the manifold $\tilde{F}_0 \subset \Gamma_0$ satisfying the following equations:

$$H_0(x, I_r; \mu) = H_0(\bar{x}_0(I_r; \mu), I_r; \mu), \quad \mathcal{H}_\varepsilon(I, \phi; \mu) = h.$$

Proof: We will prove the proposition for $W^u(\mathcal{A}_\epsilon)$ because the case of $W^s(\mathcal{A}_\epsilon)$ is analogous to that. Let us select some orbit γ_ϵ of the restricted form (4.9) with the properties as in proposition 5.2. We fix some small $\delta_0 > 0$ and introduce the notation

$$\mathcal{S}_\epsilon^u = W_{loc}^u(\mathcal{A}_\epsilon) \cup \mathcal{A}_\epsilon - \partial U^{\delta_0}.$$

Note that \mathcal{S}_ϵ^u is a three-dimensional manifold with boundary and $\partial \mathcal{S}_\epsilon^u = \mathcal{A}_\epsilon - \partial \mathcal{A}_\epsilon$.

Consider a point $p_\epsilon \in \gamma_\epsilon$. Using proposition 5.2 we immediately see that $E_\epsilon(h)$ transversally intersects $\partial \mathcal{S}_\epsilon^u$ at p_ϵ . Since $T_{p_\epsilon}(\partial \mathcal{S}_\epsilon^u) \subset T_{p_\epsilon} \mathcal{S}_\epsilon^u$, we conclude that $E_\epsilon(h)$ is transversal to \mathcal{S}_ϵ^u itself at p_ϵ . Since transversality can be expressed as a rank condition on the inclusion maps of $E_\epsilon(h)$ and \mathcal{S}_ϵ^u , it must hold in some $\mathcal{B}_{p_\epsilon}^u$ (relatively) open subset of \mathcal{S}_ϵ^u with $p_\epsilon \in \mathcal{B}_{p_\epsilon}^u$. Thus $E_\epsilon(h)$ has a nonempty transversal intersection with $\mathcal{B}_{p_\epsilon}^u$, which proves (i). Counting the dimensions of the two manifolds we conclude that $E_\epsilon(h) \cap \mathcal{B}_{p_\epsilon}^u$ is a two-dimensional manifold with boundary thus $E_\epsilon(h)$ intersects \mathcal{S}_ϵ^u locally transversally in a two-dimensional manifold \mathcal{G}_ϵ^u with $\partial \mathcal{G}_\epsilon^u \subset \gamma_\epsilon$. This construction can be performed for any $p_\epsilon \in \gamma_\epsilon$, thus we can construct a two-dimensional manifold \mathcal{G}_ϵ^u with $\partial \mathcal{G}_\epsilon^u = \gamma_\epsilon$. Now \mathcal{G}_ϵ^u is a locally transversal intersection of \mathcal{S}_ϵ^u with $E_\epsilon(h)$ but, by remark 5.1, it is also a globally transversal intersection. Moreover, since $\mathcal{G}_\epsilon^u \subset W^u(\mathcal{A}_\epsilon) \cup \mathcal{A}_\epsilon$ and $\mathcal{G}_\epsilon^u \cap \mathcal{A}_\epsilon = \gamma_\epsilon$, we have $W^u(\gamma_\epsilon) = \mathcal{G}_\epsilon^u$, which completes the proof of (ii).

Decreasing δ_0 , if necessary, we obtain from proposition 4.2 that γ_ϵ consists of the base-points of a smooth one parameter family F_ϵ^s of perturbed stable fibers, all having the same energy. This implies that $W_{loc}^s(\gamma_\epsilon) = F_\epsilon^s$ (but it does not mean that the fibers in F_ϵ^s are invariant under the dynamics of the perturbed standard form). We also know from (ii) of proposition 4.2 that the two-dimensional manifold F_ϵ^s is C^{r-1} ϵ -close to a two-dimensional submanifold F_0^s of Γ_0 which contains a one-parameter family of unperturbed stable fibers. Since the unperturbed fibers contain trajectories of the form (3.2), by hypothesis (H2), sufficiently close to \mathcal{A}_0 they can be approximated with arbitrary small error by curves in Γ_0 with $I = \text{constant}$, $\phi = \text{constant}$ ($D_I H_0$ in (3.2) tends to zero exponentially fast in a neighborhood of \mathcal{A}_0). Therefore, if $\delta_0 > 0$ is sufficiently small, specifying a one-parameter family \tilde{F}_0^s of curves in $\Gamma_0 \cap U^{\delta_0}$ with $I = \text{constant}$, $\phi = \text{constant}$, such that $\mathcal{H}_\epsilon(I, \phi; \mu) = h$, we can approximate F_ϵ^s with an error of $\mathcal{O}(\delta_0^2, \epsilon)$ as claimed in (iii) of the proposition (see fig. 4). An $\mathcal{O}(\epsilon)$ error comes from the fact that we use Γ_0 instead of $W_{loc}^s(\mathcal{A}_\epsilon)$, and an $\mathcal{O}(\delta_0^2)$ error comes from the fact that we consider the fibers as curves with $I = \text{constant}$, $\phi = \text{constant}$ within U^{δ_0} . Clearly, the same reasoning applies in the case of $W_{loc}^s(\gamma_\epsilon)$ yielding the set \tilde{F}_0^u , which concludes the proof of (iii) with $\tilde{F}_0 = \tilde{F}_0^s \cup \tilde{F}_0^u$. \square

Remark 5.2. In proposition 5.3 we have only asserted that $W^s(\gamma_\epsilon) \subset E_\epsilon(h) \cap W^s(\mathcal{A}_\epsilon)$ and $W^u(\gamma_\epsilon) \subset E_\epsilon(h) \cap W^u(\mathcal{A}_\epsilon)$. In particular, we did not claim that e.g. $W^s(\gamma_\epsilon) = E_\epsilon(h) \cap W^s(\mathcal{A}_\epsilon)$. In fact, $E_\epsilon(h) \cap W^s(\mathcal{A}_\epsilon)$ can have several connected components and precisely one of these connected components is equal to $W^s(\gamma_\epsilon)$.

Remark 5.3. If $\gamma_\epsilon \in \mathcal{A}_\epsilon$ is an orbit homoclinic to some equilibrium $q_\epsilon \in \mathcal{A}_\epsilon$ then by $W^s(\gamma_\epsilon)$ (respectively $W^u(\gamma_\epsilon)$) we mean the connected component of $W^s(q_\epsilon)$ (respectively $W^u(q_\epsilon)$) in $\tilde{\mathcal{P}}$ which contains γ_ϵ .

5.2. The intersection of $W^s(\mathcal{A}_\epsilon)$ and $W^u(\mathcal{A}_\epsilon)$

Our goal in this section will be to establish the existence of some “weakly recurrent” dynamics in a neighborhood of \mathcal{A}_ϵ . By recurrent we mean the existence of solutions which are contained in $W_{loc}^u(\mathcal{A}_\epsilon)$

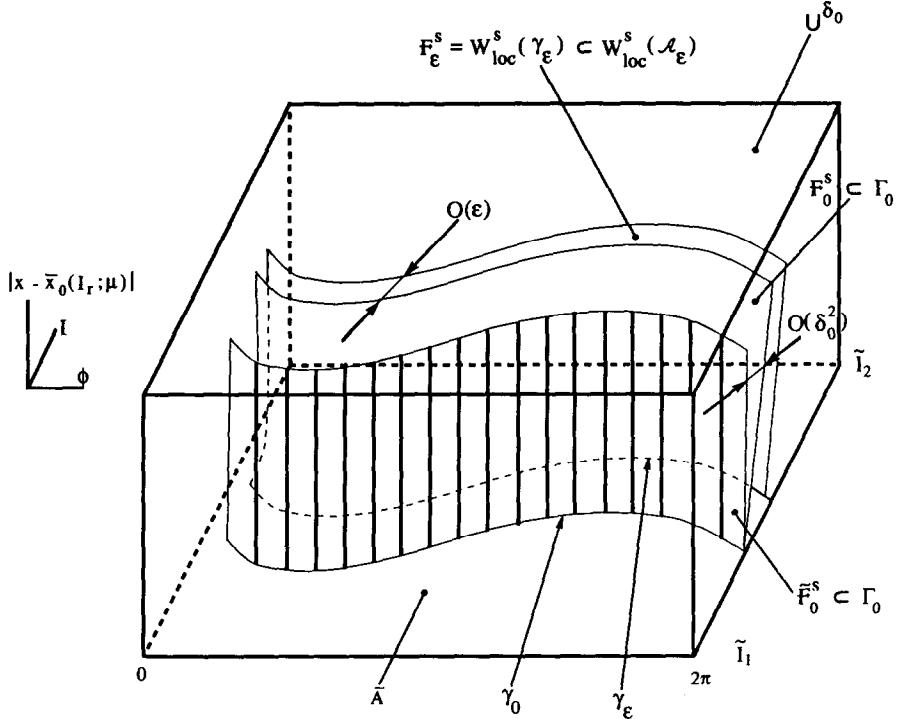


Fig. 4. The approximation of $F_\epsilon^s = W_{loc}^s(\gamma_\epsilon)$ by $\tilde{F}_0^s \subset \Gamma_0$.

and hence leave a neighborhood of \mathcal{A}_ϵ , but return after some time to the same neighborhood, as they are also contained in $W^s(\mathcal{A}_\epsilon)$. The weakness of this recurrence comes from the fact, that $W_{loc}^s(\mathcal{A}_\epsilon)$ and $W_{loc}^u(\mathcal{A}_\epsilon)$ are only locally invariant (see proposition 4.2) and a trajectory, at least a priori, may leave them as $t \rightarrow \pm\infty$.

The usual way to study the intersection of stable and unstable manifolds of a normally hyperbolic invariant manifold is the higher-dimensional Melnikov-analysis (see e.g. Wiggins [19]). In particular, Holmes and Marsden [8] study the intersection of invariant manifolds in two-degree-of-freedom Hamiltonian systems (see also Guckenheimer and Holmes [23] and Wiggins [19]). In all these cases the analysis proceeds as follows:

- Step 1. Explore the geometry of the unperturbed normally hyperbolic manifold \mathcal{M}_0 and its stable and unstable manifolds, $W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$. Assume that the stable and unstable manifolds have a common connected component Γ_0 .
- Step 2. Find some nontrivial recurrent set \mathcal{S}_0 (e.g. periodic solution, n -torus) within \mathcal{M}_0 that smoothly persists under small perturbation, and locate $W^s(\mathcal{S}_0)$ and $W^u(\mathcal{S}_0)$ within $W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$.
- Step 3. Perturb this structure and measure the splitting of $W^s(\mathcal{S}_\epsilon)$ and $W^u(\mathcal{S}_\epsilon)$ in a coordinate system attached to Γ_0 . Using an idea of Melnikov [9] for time periodic perturbations of planar systems, derive a computable expression (usually called the *Melnikov-function*) for the leading order term of the distance of $W^s(\mathcal{S}_\epsilon)$ and $W^u(\mathcal{S}_\epsilon)$.
- Step 4. Find the zeros of the Melnikov function and use the implicit function theorem to show that it indeed indicates an intersection of $W^s(\mathcal{S}_\epsilon)$ and $W^u(\mathcal{S}_\epsilon)$, and give conditions for the intersection to be transversal.

In our case, step 1 was discussed in section 3 with $\mathcal{M}_0 = \mathcal{A}_0$, but we cannot perform step 2 for the following reason: our \mathcal{A}_0 consists of equilibria most of which (with the possible exception of a finite number of them) will be destroyed under small generic perturbations in the standard form. We will therefore skip step 2 and carry out step 3 and step 4, but with \mathcal{A}_ϵ instead of \mathcal{S}_ϵ , i.e. we will look at the intersection of $W^s(\mathcal{A}_\epsilon)$ and $W^u(\mathcal{A}_\epsilon)$ directly. Technically speaking, this is an easier problem since we have to determine the splitting of hypersurfaces in $\tilde{\mathcal{P}}$ and measure their distance along only one direction, which is transversal to both of them. At the same time we will not get as strong results as in the standard Melnikov analysis, because we will not be able to predict the existence of trajectories which stay in $W^s(\mathcal{A}_\epsilon) \cap W^u(\mathcal{A}_\epsilon)$ for all times. (Later, we will be able to make predictions like this, but we will need the idea of fibering of $W_{loc}^s(\mathcal{A}_\epsilon) \cap W_{loc}^u(\mathcal{A}_\epsilon)$, as it was explained in section 4.2.)

Let us first define the energy-difference function

$$\begin{aligned}\Delta\mathcal{H}(\phi; \mu) &= \mathcal{H}(\eta, \phi; \mu) - \mathcal{H}(\eta, \phi - \Delta\phi(\mu); \mu), \\ &= H_p(\bar{x}_0(I_r; \mu), I_r, \phi; \mu) - H_p(\bar{x}_0(I_r; \mu), I_r, \phi - \Delta\phi(\mu); \mu)\end{aligned}\quad (5.5)$$

(with $\Delta\phi(\mu)$ defined in (3.2)), and the set

$$Z_\mu^+ = \{(\eta, \phi) \in A \mid \Delta\mathcal{H}(\phi; \mu) = 0, D_\phi \Delta\mathcal{H}(\phi; \mu) \neq 0\}, \quad (5.6)$$

which is clearly the set of transversal zeros of $\Delta\mathcal{H}$ in A . Note that if Z_μ^+ is not empty then it consists of a set of $\phi = \text{constant}$ lines. This is a feature, however, which will not necessarily hold in the case of assumption (H2').

We will now formulate a result concerning the intersection of $W^s(\mathcal{A}_\epsilon)$ and $W^u(\mathcal{A}_\epsilon)$ in terms of $\Delta\mathcal{H}$ and $\Delta\phi$. We will not spell out all the lengthy details which are in common with the standard Melnikov analysis, but rather try to focus on the differences. One of these differences is the fact that our unperturbed standard form (2.5)₀ is usually not Hamiltonian. We remark that Kovačić [24] used a similar construction to measure the splitting of the stable and unstable manifolds in a general dissipative context. The Hamiltonian nature of our problem brings a simplification compared to his results and our assertions are different from his. The reader may also consult Wiggins [19] for details of related calculations.

Lemma 5.4. Let us suppose that there exists $\bar{\mu} \in V$ such that $Z_{\bar{\mu}}^+$ is not empty and $p = (\bar{\eta}, \bar{\phi}) \in Z_{\bar{\mu}}^+$. Then, there exists ϵ_0 such that if $\epsilon < \epsilon_0$ then

- (i) $W^s(\mathcal{A}_\epsilon) \cap W^u(\mathcal{A}_\epsilon) = \Gamma_\epsilon \neq \emptyset$.
- (ii) The intersection of $W^s(\mathcal{A}_\epsilon)$ and $W^u(\mathcal{A}_\epsilon)$ is transversal along an orbit y_ϵ of the standard form.
- (iii) There exists $\delta_0 > 0$ and a trajectory $y_0(t, I_r + \sqrt{\epsilon}\bar{\eta}, \phi_0; \bar{\mu})$ of (2.5)₀ of the form (3.2), such that $y_0(t, I_r + \sqrt{\epsilon}\bar{\eta}, \phi_0; \bar{\mu})$ is ϵ -close to y_ϵ outside of U^{δ_0} and

$$\begin{aligned}y_0(+\infty, I_r + \sqrt{\epsilon}\bar{\eta}, \phi_0; \bar{\mu}) &= \begin{pmatrix} \bar{x}_0(I_r; \bar{\mu}) \\ I_r + \sqrt{\epsilon}\bar{\eta} \\ \bar{\phi} \end{pmatrix}, \\ y_0(-\infty, I_r + \sqrt{\epsilon}\bar{\eta}, \phi_0; \bar{\mu}) &= \begin{pmatrix} \bar{x}_0(I_r; \bar{\mu}) \\ I_r + \sqrt{\epsilon}\bar{\eta} \\ \bar{\phi} - \Delta\phi(\bar{\mu}) \end{pmatrix}.\end{aligned}\quad (5.7)$$

(See fig. 5 for illustration.)

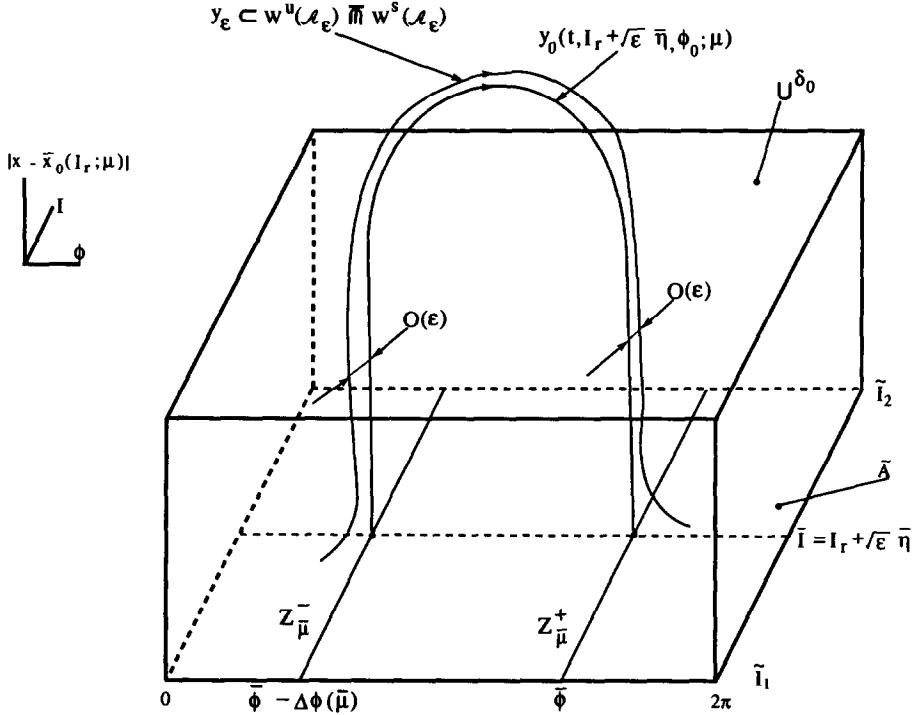


Fig. 5. Statement (iii) of lemma 5.4.

(iv) ∂U^{δ_0} has an open neighborhood N^{δ_0} such that $\Gamma_\epsilon \cap N^{\delta_0}$ is C^{r-1} δ_0^2 -close to a manifold $\tilde{\Gamma}_0 \cap N^{\delta_0}$ where $\tilde{\Gamma}_0 \subset \Gamma_0$ satisfies

$$H_0(x, I_r; \bar{\mu}) = H_0(\bar{x}_0(I_r; \bar{\mu}), I_r; \bar{\mu}), \quad \Delta \mathcal{H}(\phi; \bar{\mu}) = 0.$$

(v) (i)–(iv) are also satisfied for μ values sufficiently close to $\bar{\mu}$.

Proof: We first note that since (2.5a)₀ is a Hamiltonian system, Γ_0 satisfies the equation

$$H_0(x, I_r; \mu) = H_0(\bar{x}_0(I_r; \mu), I_r; \mu).$$

This, together with the parameterization of γ_0 in (3.1) shows that a normal to Γ_0 at a point described by the parameters (t_0, I, ϕ_0) is given by

$$n(-t_0, I, \phi_0) = DH_0(x^h(-t_0, I_r; \mu), I_r; \mu),$$

where D again refers to the gradient taken in the variables (x, I, ϕ) . For small $\epsilon > 0$ both $W^s(\mathcal{A}_\epsilon)$ and $W^u(\mathcal{A}_\epsilon)$ intersect n transversally in two points, y_ϵ^s and y_ϵ^u , respectively, which are locally unique and may coincide. We define the signed splitting distance of $W^s(\mathcal{A}_\epsilon)$ and $W^u(\mathcal{A}_\epsilon)$ at (t_0, I, ϕ_0) as

$$d(t_0, I, \phi_0; \mu, \epsilon) = \frac{\langle n(-t_0, I, \phi_0; \mu), y_\epsilon^s - y_\epsilon^u \rangle}{\|n(-t_0, I, \phi_0; \mu)\|}.$$

It can be shown (see the references mentioned above) that d can be written as

$$d(t_0, I, \phi_0; \mu, \varepsilon) = \varepsilon \frac{M(t_0, I, \phi_0; \mu)}{\|n(-t_0, I, \phi_0; \mu)\|} + \mathcal{O}(\varepsilon^2), \quad (5.8)$$

where M is the *Melnikov-function* given by

$$M(t_0, I, \phi_0; \mu) = \int_{-\infty}^{+\infty} \{H_0, H_p\}|_{y_0(t-t_0, I, \phi_0; \mu)} dt = \int_{-\infty}^{+\infty} \frac{d}{dt} H_p\Big|_{y_0(t-t_0, I, \phi_0; \mu)} dt, \quad (5.9)$$

with the solution y_0 in the form as in (3.2) and $\{ , \}$ denoting the canonical Poisson-bracket. Evaluating (5.9) we find that

$$\begin{aligned} M(t_0, I, \phi_0; \mu) &= H_p|_{y_0(+\infty, I, \phi_0; \mu)} - H_p|_{y_0(-\infty, I, \phi_0; \mu)} \\ &= H_p(\bar{x}_0(I_r, \mu), I_r + \sqrt{\varepsilon}\bar{\eta}, \bar{\phi}; \mu) - H_p(\bar{x}_0(I_r, \mu), I_r + \sqrt{\varepsilon}\bar{\eta}, \bar{\phi} - \Delta\phi(\mu); \mu), \end{aligned} \quad (5.10)$$

where we have introduced the notation

$$\bar{\phi} = \phi_0 + \int_{t_0}^{+\infty} D_I H_0(x^h(t, I_r; \mu), I_r; \mu) dt. \quad (5.11)$$

For small ε we Taylor-expand (5.10) to obtain

$$M(t_0, I, \phi_0; \mu) = \Delta\mathcal{H}(\bar{\phi}; \mu) + \mathcal{O}(\sqrt{\varepsilon}). \quad (5.12)$$

Using (5.8) and the implicit function theorem we see that if $D_\phi \Delta\mathcal{H}(\bar{\phi}; \mu) \neq 0$ then for $\mu = \bar{\mu}$, small positive ε and any $I = \bar{I} = I_r + \sqrt{\varepsilon}\bar{\eta} \in [\tilde{I}_1, \tilde{I}_2]$, the distance function in (5.8) has a transverse zero $\mathcal{O}(\varepsilon)$ -close to $(t_0(\bar{\phi}), \bar{I}, \phi_0(\bar{\phi}))$. This means that $W^s(\mathcal{A}_\varepsilon)$ and $W^u(\mathcal{A}_\varepsilon)$ intersect transversally in an orbit y_ε which has a point $\mathcal{O}(\varepsilon)$ -close to the point of Γ_0 with parameters $(\bar{t}_0, \bar{I}, \bar{\phi}_0)$. Based on this, (5.5) and (5.10)–(5.12) imply (i) and (ii) of the lemma. It also follows from the standard Melnikov theory, that the solution $y_0(t - t_0, I, \phi_0; \mu)$ used in (5.4) is $\mathcal{O}(\varepsilon)$ -close to y_ε outside some fixed small neighborhood of \mathcal{A}_ε , which proves (iii). By construction, the zero set of the Melnikov function is a subset Z_0 of Γ_0 . Z_0 has the property that it is C'^{-1} ε -close to Γ_ε outside U^{δ_0} provided ε is small. We will show that the set $\tilde{\Gamma}_0$, defined in (iv) of the lemma, can be used as a local approximation for Z_0 . The advantage of using $\tilde{\Gamma}_0$ instead of Z_0 to approximate Γ_ε is that $\tilde{\Gamma}_0$ is defined in terms of the coordinates of the phase space, i.e. independent of the parameterization of Γ_0 .

Choosing δ_0 small enough, we can make the difference $\bar{\phi} - \phi_0$ to be of $\mathcal{O}(\delta_0^2)$, as it was pointed out in the proof of proposition 5.3. Hence in a neighborhood N^{δ_0} of U^{δ_0} the set Z_0 satisfies

$$M(t_0, I, \phi_0) = \Delta\mathcal{H}(\phi_0 + \nu(t_0); \mu) = 0,$$

where $\nu(t_0) = \mathcal{O}(\delta_0^2)$. By the implicit function theorem, if ϕ_0 is a transverse zero of $\Delta\mathcal{H}(\phi; \mu)$ then an

open subset of Z_0 satisfies

$$\phi = \phi_0 + \mathcal{O}(\delta_0^2), \quad H_0(x, I_r; \mu) = H_0(\bar{x}_0(I_r; \mu), I_r; \mu),$$

from which (iv) follows. Finally, a standard application of the implicit function theorem proves (v). \square

Remark 5.4. The Melnikov-function in (5.9) is formally the same as the one obtained in Holmes and Marsden [8] for a class of two-degree-of-freedom Hamiltonian systems, however, it has a very different meaning. Their Melnikov-function gives a criterion for the existence of transverse homoclinic orbits of such periodic orbits which are bounded away from resonances and completely encircle the annulus. Our study will be concerned with internal orbits, some of which are periodic orbits *created* by the perturbation that *do not* encircle the annulus.

5.3. Orbits asymptotic to internal orbits: the energy-phase criterion

In this section we will put together all the results we have listed in sections 4 and 5. Our ultimate goal is to gain some understanding of the dynamics of the perturbed system $(2.1)_\epsilon$ near the manifold \mathcal{A}_ϵ . It is important to note that we will only use three ingredients in formulating our results: 1. Our knowledge about the unperturbed geometry of the standard form, embodied in the phase shift $\Delta\phi$. 2. The dynamics of the reduced system. 3. The structure of the transverse zero set Z_μ^+ of the energy difference function $\Delta\mathcal{H}$, and the structure of the set

$$Z_\mu^- = \{(\eta, \phi) \in A \mid (\eta, \phi + \Delta\phi(\mu)) \in Z_\mu^+ \}, \quad (5.13)$$

which is just the counterpart of Z_μ^+ in the following sense: trajectories of $(2.5)_0$, which are asymptotic to a point of Z_μ^+ in forward time, are asymptotic to the corresponding point in Z_μ^- in backward time. We remark that $Z_\mu^+ \neq \emptyset$ implies $Z_\mu^- \neq \emptyset$. Finally, we emphasize that *all the distances mentioned in theorem 5.5 are measured in the coordinates (x, I, ϕ) .*

Theorem 5.5. Let us assume that (H1) and (H2) hold, and (see fig. 6)

- (A1) There exists $\bar{\mu} \in V$ such that $Z_{\bar{\mu}}^+ \neq \emptyset$,
- (A2) γ_0^+ and γ_0^- are internal orbits of the reduced system (4.10) with $\gamma_0^+ \cap Z_{\bar{\mu}}^+ = p_0^+ = (\bar{\eta}, \bar{\phi}) \in A$ and $\gamma_0^- \cap Z_{\bar{\mu}}^- = p_0^- = (\bar{\eta}, \bar{\phi} - \Delta\phi(\bar{\mu})) \in A$,

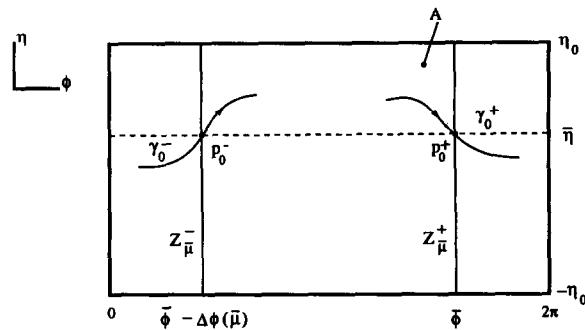


Fig. 6. Visualization of assumptions (A1)–(A3) in theorem 5.5.

(A3) γ_0^+ intersects $Z_{\bar{\mu}}^+$ transversally at p_0^+ and γ_0^- intersect $Z_{\bar{\mu}}^-$ transversally at p_0^- .

Then, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the following are satisfied:

- (i) The perturbed system $(2.1)_\varepsilon$ has two internal orbits, $\gamma_\varepsilon^+, \gamma_\varepsilon^- \in \mathcal{A}_\varepsilon$ which are $C^{r-1} \sqrt{\varepsilon}$ -close in A to γ_0^+ and γ_0^- , respectively.
 - (ii) The perturbed system $(2.1)_\varepsilon$ also has two internal orbits, $\tilde{\gamma}_\varepsilon^+, \tilde{\gamma}_\varepsilon^- \subset \mathcal{A}_\varepsilon$ such that $\tilde{\gamma}_\varepsilon^+$ intersects $Z_{\bar{\mu}}^+$ transversally at \tilde{p}_ε^+ with $d(\tilde{p}_\varepsilon^+, p_0^+) = O(\sqrt{\varepsilon})$ in A , and $\tilde{\gamma}_\varepsilon^-$ intersects $Z_{\bar{\mu}}^-$ transversally at \tilde{p}_ε^- with $d(\tilde{p}_\varepsilon^-, p_0^-) = O(\sqrt{\varepsilon})$ in A . Moreover, $W^s(\gamma_\varepsilon^+)$ intersects $W^u(\tilde{\gamma}_\varepsilon^-)$ and $W^u(\gamma_\varepsilon^-)$ intersects $W^s(\tilde{\gamma}_\varepsilon^+)$ in *transverse heteroclinic orbits* y_ε^1 and y_ε^2 , respectively (see fig. 7a).
 - (iii) There exists $\delta_0 > 0$ and a trajectory $y_0(t, I_r + \sqrt{\varepsilon}\bar{\eta}, \phi_0; \bar{\mu})$ of $(2.5)_0$ of the form (3.2) such that $y_0(t, I_r + \sqrt{\varepsilon}\bar{\eta}, \phi_0; \bar{\mu})$ is $O(\varepsilon)$ -close to the orbits y_ε^1 and y_ε^2 outside of U^{δ_0} and
- $$y_0(+\infty, I_r + \sqrt{\varepsilon}\bar{\eta}, \phi_0; \bar{\mu}) = p_0^+, \quad y_0(-\infty, I_r + \sqrt{\varepsilon}\bar{\eta}, \phi_0; \bar{\mu}) = p_0^-,$$
- as seen in fig. 7a.
- (iv) If $\gamma_0^+ = \gamma_0^-$ in (A2) then $\gamma_\varepsilon^+ = \tilde{\gamma}_\varepsilon^+ = \gamma_\varepsilon^- = \tilde{\gamma}_\varepsilon^-$, and $y_\varepsilon^1 = y_\varepsilon^2$ is a *transverse homoclinic orbit* to γ_ε^+ , such that (iii) still holds (see fig. 7b).
 - (v) (i)–(iv) are also satisfied for μ values sufficiently close to $\bar{\mu}$.

Proof: We will prove the theorem for the standard form for convenience, which directly implies the same results for the perturbed system near the resonance. Statement (i) immediately follows from proposition 5.2 so we turn to the proof of (ii).

Using assumptions (A1)–(A3) and the definition of $Z_{\bar{\mu}}^+$ and $Z_{\bar{\mu}}^-$, we can appeal to lemma 5.4 to infer that $W^s(\mathcal{A}_\varepsilon)$ intersects $W^u(\mathcal{A}_\varepsilon)$ transversally in a two-dimensional manifold Γ_ε . By construction, $Z_{\bar{\mu}}^+$ contains the basepoints of unperturbed stable fibers which form the manifold $\tilde{\Gamma}_0$ of (iv) of lemma 5.4. By

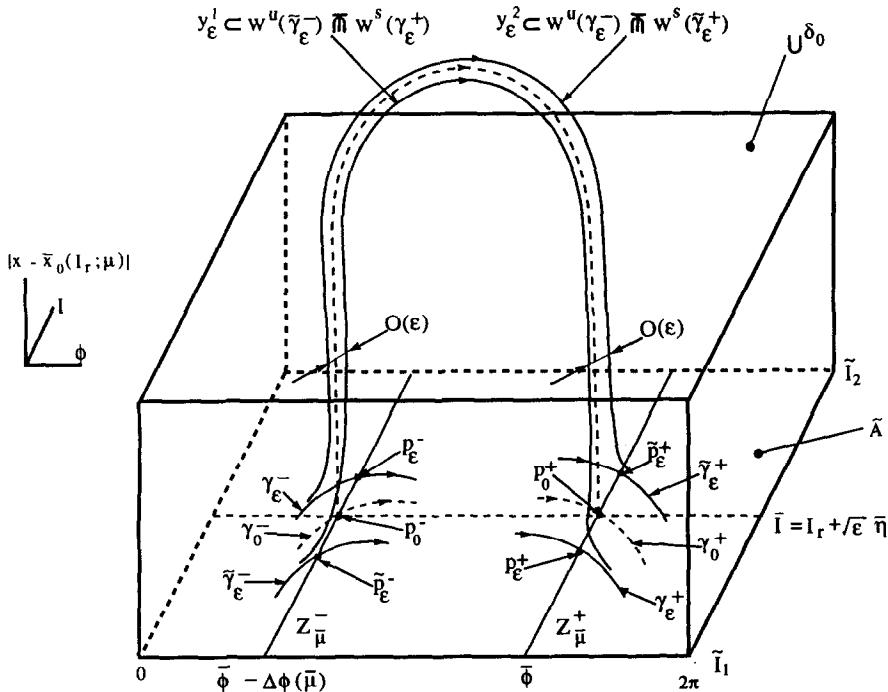


Fig. 7a. Statement (ii) of theorem 5.5.

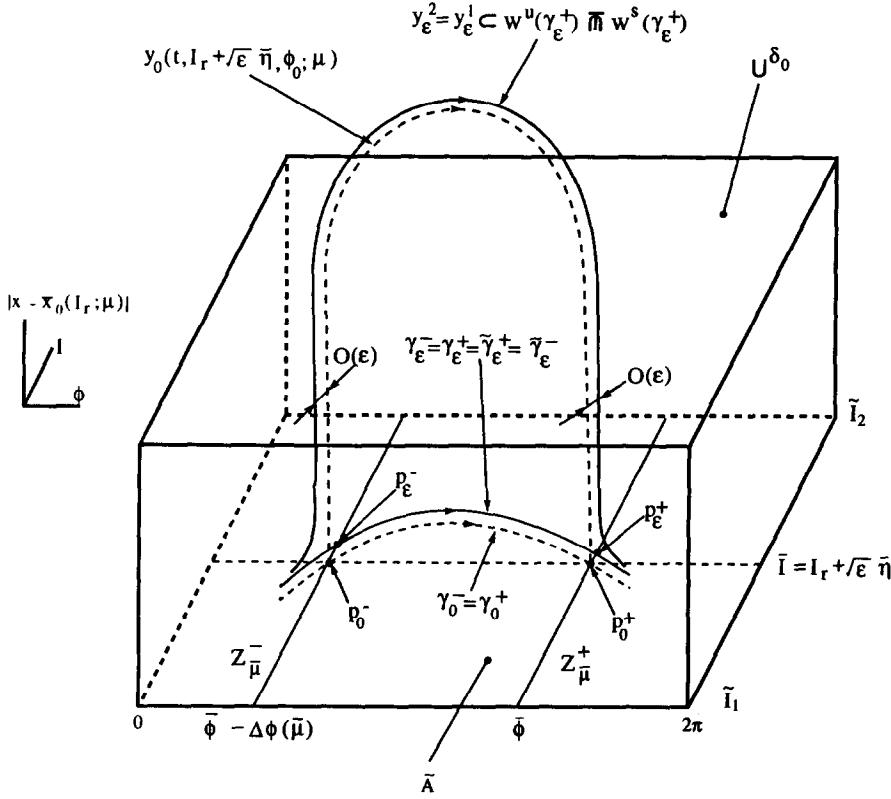


Fig. 7b. Statement (iv) of theorem 5.5.

(ii) of proposition 4.2 and remark 4.2, the basepoints of the stable fibers contained in Γ_ϵ form a set $Z_{\bar{\mu}, \epsilon}^+$ which is $C^{r-1} \sqrt{\epsilon}$ -close to $Z_{\bar{\mu}}^+$, hence intersects γ_ϵ^+ transversally $\mathcal{O}(\sqrt{\epsilon})$ -close to p_0^+ . By (iv) of proposition 4.2 this means that Γ_ϵ contains an orbit y_ϵ^1 which is positively asymptotic to γ_ϵ^+ . Also, by lemma 5.4, there exists an unperturbed trajectory $y_0(t, I_r + \sqrt{\epsilon}\bar{\eta}, \phi_0; \bar{\mu})$ satisfying statement (iii) of this theorem in relation with y_ϵ^1 . We know that $y_\epsilon^1 \subset \Gamma_\epsilon$, hence it intersects $\partial U^{\delta_0} \cap W_{loc}^u(\mathcal{A}_\epsilon)$ $\mathcal{O}(\epsilon)$ -close to the intersection of $y_0(t, I_r + \sqrt{\epsilon}\bar{\eta}, \phi_0; \bar{\mu})$ with $\partial U^{\delta_0} \cap W_{loc}^u(\mathcal{A}_0)$. Again, using remark 4.2, we conclude the existence of an internal orbit $\tilde{\gamma}_\epsilon^- \subset \mathcal{A}_\epsilon$ containing the basepoint \tilde{p}_ϵ^- of an unstable perturbed fiber $f_\epsilon^u(\tilde{p}_\epsilon^-)$ with $d(\tilde{p}_\epsilon^-, p_0^-) = \mathcal{O}(\sqrt{\epsilon})$, and y_ϵ^1 intersects $f_\epsilon^u(\tilde{p}_\epsilon^-)$. By (iv) of proposition 4.2 and by proposition 5.2 this implies that $y_\epsilon^1 \subset W^s(\gamma_\epsilon^+) \cap W^u(\tilde{\gamma}_\epsilon^-)$. We can repeat the same construction in reverse time for γ_0^- to establish the existence of a heteroclinic orbit y_ϵ^1 and an internal orbit $\tilde{\gamma}_\epsilon^+$ such that $y_\epsilon^2 \subset W^u(\gamma_\epsilon^-) \cap W^s(\tilde{\gamma}_\epsilon^+)$. We also obtain that the statement regarding y_ϵ^2 in (iii) is satisfied. Hence, we have proved (ii) and (iii) with the exception of showing that y_ϵ^1 and y_ϵ^2 are in fact *transverse* heteroclinic orbits, which we now turn to. We will prove transversality for y_ϵ^1 , since the case of y_ϵ^2 is completely analogous.

We will first show that if $H|_{y_\epsilon^1} = h$, then $E_\epsilon(h)$ intersects Γ_ϵ transversally in y_ϵ^1 . If $q_\epsilon = y_\epsilon^1 \cap \partial U^{\delta_0} \cap W_{loc}^s(\mathcal{A}_\epsilon)$, then it suffices to prove that $T_{q_\epsilon}(E_\epsilon(h) \cap W^s(\mathcal{A}_\epsilon))$ has a one-dimensional subspace not contained in $T_{q_\epsilon}(\Gamma_\epsilon) = T_{q_\epsilon}(\Gamma_\epsilon \cap W^s(\mathcal{A}_\epsilon))$. According to proposition 5.3, for fixed sufficiently small $\delta_0 > 0$ and $\epsilon > 0$ with $0 < \epsilon \ll \delta_0 \ll 1$, $E_\epsilon(h) \cap W^s(\mathcal{A}_\epsilon)$ is locally $\mathcal{O}(\epsilon, \delta_0^2)$ -close to a manifold satisfying

$$H_0(x, I_r; \mu) = H_0(\bar{x}_0(I_r; \mu), I_r; \mu), \quad \mathcal{H}_\epsilon(I, \phi; \mu) = h. \quad (5.14a)$$

By lemma 5.4, Γ_ϵ is locally approximated with an error of $\mathcal{O}(\epsilon, \delta_0^2)$ by a manifold satisfying

$$H_0(x, I_r; \bar{\mu}) = H_0(\bar{x}_0(I_r, \bar{\mu}), I_r; \bar{\mu}), \quad \Delta\mathcal{H}(\phi; \bar{\mu}) = 0. \quad (5.14b)$$

In accordance with our discussion above, we have to show that the map

$$G_\epsilon : N^{\delta_0} \rightarrow \mathbb{R}^3, \quad \begin{pmatrix} x \\ I \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} H_0(x, I_r; \bar{\mu}) - H_0(\bar{x}_0(I_r, \bar{\mu}), I_r; \bar{\mu}) \\ \Delta\mathcal{H}(\phi; \bar{\mu}) \\ \mathcal{H}_\epsilon(I, \phi; \bar{\mu}) - h \end{pmatrix},$$

is a submersion around $q_0 = q_\epsilon + \mathcal{O}(\epsilon, \delta_0^2)$, as in the proof of proposition 5.2. We compute DG_ϵ to obtain

$$DG_\epsilon = \begin{pmatrix} D_x H_0 & 0 & 0 \\ 0 & 0 & D_\phi \Delta\mathcal{H} \\ 0 & D_I \mathcal{H}_\epsilon & D_\phi \mathcal{H}_\epsilon \end{pmatrix}.$$

At q_0 , $D_x H_0 \neq 0$ and we assume that $D_{x_1} H_0|_{q_0} = \mathcal{O}(\delta_0) \neq 0$ (if $D_{x_1} H_0|_{q_0} = 0$ then we can repeat the following argument with $D_{x_2} H_0|_{q_0} = \mathcal{O}(\delta_0) \neq 0$). The determinant of the matrix L_ϵ which contains the first, third and fourth columns of DG_ϵ , can be seen to equal

$$\det(L_\epsilon) = -D_{x_1} H_0 D_\phi \Delta\mathcal{H} D_I \mathcal{H}_\epsilon = -\epsilon D_{x_1} H_0 D_\phi \Delta\mathcal{H} D_I \mathcal{H} + \mathcal{O}(\epsilon^{3/2}),$$

from which we have

$$\begin{aligned} \det(L_\epsilon)|_{q_0} &= -\sqrt{\epsilon} D_{x_1} H_0|_{q_0} [D_\phi \Delta\mathcal{H}|_{p_0^+} D_\eta \mathcal{H}|_{p_0^+} + \mathcal{O}(\sqrt{\epsilon}, \delta_0^2)] \\ &= \sqrt{\epsilon} D_{x_1} H_0|_{q_0} [\langle D\Delta\mathcal{H}, JD\mathcal{H} \rangle|_{p_0^+} + \mathcal{O}(\sqrt{\epsilon}, \delta_0^2)], \end{aligned} \quad (5.15)$$

where, for notational simplicity, D denotes differentiation in η and ϕ . In (5.15) we used the fact that by construction, the ϕ coordinates of q_0 and p_0^+ differ by an $\mathcal{O}(\delta_0^2)$ amount, while their I coordinates may differ by an $\mathcal{O}(\sqrt{\epsilon})$ amount, since both of them are contained in the phase space \tilde{P} .

Now $D\Delta\mathcal{H}|_{p_0^+}$ is the normal of $Z_{\bar{\mu}}^+$ at p_0^+ , and $JD\mathcal{H}$ is tangent to γ_0^+ at p_0^+ , thus by assumption (A3) of the theorem, $(L_\epsilon)|_{q_0}$ is nonsingular for small $\epsilon > 0$ and sufficiently small δ_0 , as we see from (5.15). Consequently, DG_ϵ has maximal rank in a neighborhood of q_0 , implying the intersection of $E_\epsilon(h)$ with Γ_ϵ to be transversal at q_ϵ for small positive ϵ . Then, by remark 5.1, $E_\epsilon(h)$ and Γ_ϵ intersect transversally along y_ϵ^1 .

Let $\tilde{E}_\epsilon(h) \subset E_\epsilon(h)$ be an open subset of $E_\epsilon(h)$ with $\tilde{E}_\epsilon(h) \cap \Gamma_\epsilon = y_\epsilon^1$. By the above discussion, $\tilde{E}_\epsilon(h) \cap \Gamma_\epsilon = (\tilde{E}_\epsilon(h) \cap W^s(\mathcal{A}_\epsilon)) \cap (\tilde{E}_\epsilon(h) \cap W^u(\mathcal{A}_\epsilon))$ is a codimension-two submanifold of $\tilde{E}_\epsilon(h)$. From proposition 5.3 we know that $\tilde{E}_\epsilon(h) \cap W^s(\mathcal{A}_\epsilon)$ is a codimension-one submanifold of $\tilde{E}_\epsilon(h)$. Hence at any $p \in y_\epsilon^1$, $T_p(\tilde{E}_\epsilon(h) \cap W^s(\mathcal{A}_\epsilon)) = T_p(W^s(\gamma_\epsilon^+))$ contains a one-dimensional subspace which is not contained in $T_p(\tilde{E}_\epsilon(h) \cap W^u(\mathcal{A}_\epsilon)) = T_p(W^u(\tilde{\gamma}_\epsilon^-))$. This implies that

$$T_p(W^s(\gamma_\epsilon^+)) + T_p(W^u(\tilde{\gamma}_\epsilon^-)) = T_p(\tilde{E}_\epsilon(h)) = T_p(E_\epsilon(h))$$

at any $p \in y_\epsilon^1$, which concludes the proof of (ii) of the theorem.

As for (iv), we now recall that by proposition 5.2 all the orbits of the restricted system passing through a $U_{p_0^-}$ neighborhood of p_0^- have different energies, since each of them lies in a transversal intersection of its energy surface with \mathcal{A}_ϵ . Then, repeating the proof of (ii) we obtain, that \tilde{p}_ϵ^- must lie in $\tilde{\gamma}_\epsilon^- = \gamma_0^+$ from which (iv) follows. Finally, as in lemma 5.4, an application of the implicit function theorem proves (v). \square

Using the structural stability of internal orbits and the property of the fibering of the local stable and unstable manifolds of \mathcal{A}_ϵ given in (i) of proposition 4.2, we can be more specific about the heteroclinic connections established in theorem 5.5. In particular, we have the following

Corollary 5.6. Let us assume that hypotheses (H1), (H2) and assumptions (A1)–(A3) of theorem 5.5 are satisfied. Then the following hold:

- (i) If γ_0^+ and γ_0^- are distinct periodic orbits then y_ϵ^1 and y_ϵ^2 are transverse heteroclinic orbits connecting periodic solutions (see e.g. fig. 8a). If $\gamma_0^+ = \gamma_0^-$ is a periodic orbit then $y_\epsilon^1 = y_\epsilon^2$ is a transverse homoclinic orbit connecting a periodic solution to itself.
- (ii) If γ_0^+ and γ_0^- are distinct homoclinic orbits then y_ϵ^1 and y_ϵ^2 are generically transverse heteroclinic orbits connecting saddle points to periodic solutions, but in degenerate (or symmetric) cases they might be transverse heteroclinic orbits connecting saddle points. If $\gamma_0^+ = \gamma_0^-$ is a homoclinic orbit

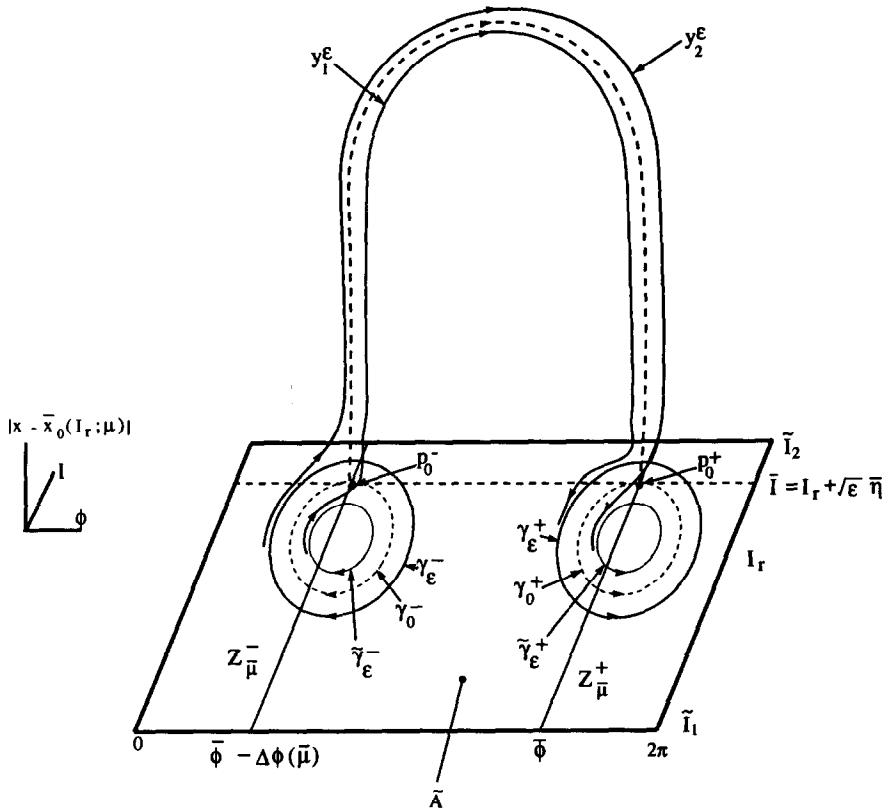


Fig. 8a. Transverse heteroclinic orbits connecting periodic solutions.

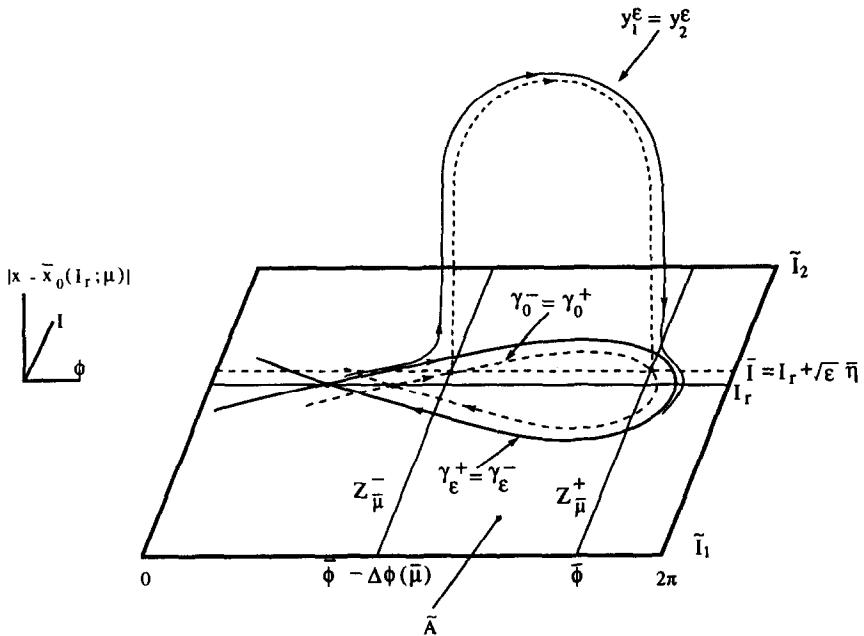


Fig. 8b. Transverse homoclinic orbit to a saddle point.

then $y_{\epsilon}^1 = y_{\epsilon}^2$ is a transverse homoclinic orbit connecting a saddle point to itself (see e.g. fig. 8b).

- (iii) If one of γ_0^+ and γ_0^- is a homoclinic orbit and the other is a periodic orbit, then one of y_{ϵ}^1 and y_{ϵ}^2 is a transverse heteroclinic orbit connecting a saddle point to a periodic solution, while the other is a transverse heteroclinic orbit connecting periodic solutions (see e.g. fig. 8c).

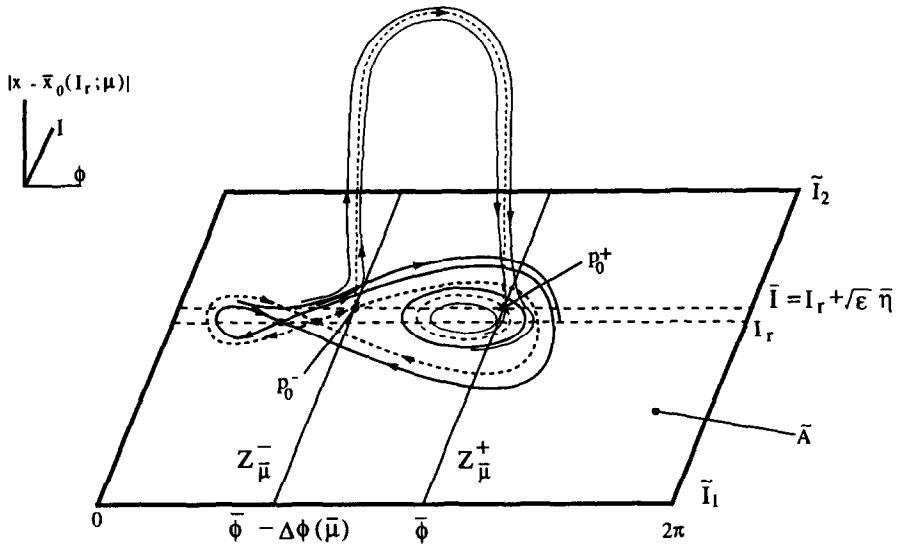


Fig. 8c. Transverse heteroclinic orbits between periodic solutions and between a saddle point and a periodic solution.

As we noted before, the sets Z_μ^+ and Z_μ^- consist of $\phi = \text{constant}$ lines in A . These lines are usually intersected by different types of internal orbits of the reduced system, so even within one problem we can obtain different kinds of connections from the variety listed in corollary 5.6. Also, we will typically obtain families of connections since an internal orbit, which intersects, say, Z_μ^+ transversally has a neighborhood filled with internal orbits transverse to Z_μ^+ . Hence via the energy-phase criterion of theorem 5.5, we can get a fairly complete understanding of the orbit structure near the resonance in the perturbed system $(2.1)_\epsilon$. We are able to predict different kinds of motions doubly asymptotic to a \mathcal{A}_ϵ , and by (i)–(iii) of the theorem, we are able to locate these connections with a good precision.

In general, the energy-phase criterion may indicate no homoclinic or heteroclinic connections for some internal orbits in a given problem. This *does not* mean that these orbits do not have transverse homoclinic orbits. If they have, however, those orbits enter and leave a small neighborhood of the given internal orbit before they approach it asymptotically. These “nonsimple” connections cannot be detected by the Melnikov-type method used in lemma 5.4.

At this point we remind the reader that the constant η_0 in (2.4) has not been specified yet. As long as $\eta_0 = \mathcal{O}(1)$, as $\epsilon \rightarrow 0$, all our results are independent of the concrete choice of η_0 . The basic principle is that one should select an η_0 value such that all orbits of the reduced system (4.10) which are bounded in ϕ are internal orbits in the annulus $A = [-\eta_0, \eta_0] \times S^1$. These are the very orbits which can be attributed to the presence of the resonance $I = I_r$. In section 7 we will see an example for such a suitable choice of η_0 .

5.4. The case of hypothesis (H2')

Throughout the previous sections we have analyzed system $(2.1)_\epsilon$ assuming that hypotheses (H1) and (H2) held. We now sketch the same procedure under hypotheses (H1) and (H2'). We will see that, although the key results are similar, there are slight differences in their formulation. Since (H2') already ensures the presence of a whole resonant manifold

$$\mathcal{A}_0 = \{(x, I, \phi) \in \mathcal{P} \mid x = \bar{x}_0(I; \mu), (I, \phi) \in A\},$$

with $A = \tilde{A} = [I_1, I_2] \times S^1$, we consider $(2.1)_\epsilon$ to be in standard form already. Notice, however, that in this case \mathcal{A}_0 has an explicit I -dependence, a feature which is responsible to a large extent for the upcoming differences in the treatment of assumption (H2'). For convenience, we let $\tilde{I}_1 = I_1$ and $\tilde{I}_2 = I_2$, and with this change of notation we can use most of our findings related to the unperturbed standard form $(2.5)_0$ for the unperturbed system $(2.1)_0$ with (H2'). One important difference is that our new phase shift

$$\Delta\phi(I; \mu) = \int_{-\infty}^{+\infty} D_I H_0(x^h(t, I; \mu), I; \mu) dt, \quad (5.16)$$

is I -dependent. Most of the phase-space structure of $(2.1)_0$ changes the same way under perturbation as it was described in section 4. In particular, we have a perturbed normally hyperbolic manifold \mathcal{A}_ϵ with locally invariant local stable and unstable manifolds discussed in section 4.1, which admit fiberings, as described in section 4.2. Lemma 4.3 applies again and yields a Hamiltonian dynamics on \mathcal{A}_ϵ . Since we have introduced no rescaling this time, our restricted Hamiltonian will be slightly different, and can be

cast in the form

$$\mathcal{H}_\varepsilon = h_0 + \varepsilon H_p(\bar{x}_0(I; \mu), I, \phi; \mu) + \mathcal{O}(\varepsilon^2),$$

where $h_0 = H_p(\bar{x}_0(I; \mu), I; \mu)$ is a constant by hypothesis (H2'). Once again, we can define the *restricted system*

$$\dot{I} = -\varepsilon D_\phi \mathcal{H}(I, \phi; \mu) + \mathcal{O}(\varepsilon^2), \quad \dot{\phi} = \varepsilon D_I \mathcal{H}(I, \phi; \mu) + \mathcal{O}(\varepsilon^2),$$

and the *reduced system*

$$I' = -D_\phi \mathcal{H}(I, \phi; \mu), \quad \phi' = D_I \mathcal{H}(I, \phi; \mu), \quad (5.17)$$

where $\mathcal{H}(I, \phi; \mu) = H_p(\bar{x}_0(I; \mu), I, \phi; \mu)$. As we indicated at the end of section 4.2, this reduced system is generally not a potential problem, but the analysis of the relation of the reduced dynamics to the “full” dynamics goes along the same lines as earlier. Lemma 5.1 still holds in the form

$$DH|_{\mathcal{A}_\varepsilon} = (-\varepsilon D_\phi \mathcal{H}(D_I \bar{x}_0)^T J, \varepsilon D_I \mathcal{H}, \varepsilon D_\phi \mathcal{H}) + \mathcal{O}(\varepsilon^2),$$

which again leads to proposition 5.2 concerning the intersection of energy surfaces containing internal orbits with \mathcal{A}_ε . Proposition 5.3 also holds, giving information about the intersection of energy surfaces with $W^s(\mathcal{A}_\varepsilon)$ and $W^u(\mathcal{A}_\varepsilon)$. Similarly to (5.5), we can define the energy-difference function

$$\Delta \mathcal{H}(I, \phi; \mu) = H_p(\bar{x}_0(I; \mu), I, \phi; \mu) - H_p(\bar{x}_0(I; \mu), I, \phi - \Delta\phi(I; \mu); \mu),$$

with $\Delta\phi(I; \mu)$ defined in (5.16). The transverse zeros of $\Delta \mathcal{H}$ constitute a set

$$Z_\mu^+ = \{(I, \phi) \in A \mid \Delta \mathcal{H}(I, \phi; \mu) = 0, D_{(I, \phi)} \Delta \mathcal{H}(I, \phi; \mu) \neq (0, 0)\}.$$

As we remarked earlier, this zero set is not necessarily a union of $\phi = \text{constant}$ lines: it is generally a union of smooth nonintersecting curves. The Melnikov-analysis of lemma 5.4 works the same way as for the case of hypothesis (H1) but we have to substitute \bar{I} for $I_r + \varepsilon \bar{\eta}$. We again define the counterpart of Z_μ^+ by

$$Z_\mu^- = \{(I, \phi) \in A \mid (I, \phi + \Delta\phi(I; \mu)) \in Z_\mu^+\},$$

and obtain the following

Theorem 5.7. Let us assume that (H1) and (H2') hold, and

- (A1) There exists $\bar{\mu} \in V$ such that $Z_{\bar{\mu}}^+ \neq \emptyset$,
- (A2) γ_0^+ and γ_0^- are internal orbits of the reduced system (4.16) with $\gamma_0^+ \cap Z_{\bar{\mu}}^+ = p_0^+ = (\bar{I}, \bar{\phi}) \in A$ and $\gamma_0^- \cap Z_{\bar{\mu}}^- = p_0^- = (\bar{I}, \bar{\phi} - \Delta\phi(\bar{I}; \bar{\mu})) \in A$,
- (A3) γ_0^+ intersects $Z_{\bar{\mu}}^+$ transversally at p_0^+ and γ_0^- intersect $Z_{\bar{\mu}}^-$ transversally at p_0^- .

Then, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the following are satisfied:

- (i) The perturbed system (2.1) $_\varepsilon$ has two internal orbits, $\gamma_\varepsilon^+, \gamma_\varepsilon^- \subset \mathcal{A}_\varepsilon$ which are C^{r-1} ε -close in A to γ_0^+ and γ_0^- , respectively.

- (ii) The perturbed system $(2.1)_\epsilon$ also has two internal orbits, $\tilde{\gamma}_\epsilon^+$, $\tilde{\gamma}_\epsilon^- \subset \mathcal{A}_\epsilon$ such that $\tilde{\gamma}_\epsilon^+$ intersect Z_μ^+ transversally at \tilde{p}_ϵ^+ with $d(\tilde{p}_\epsilon^+, p_0^+) = \mathcal{O}(\epsilon)$ in A , and $\tilde{\gamma}_\epsilon^-$ intersect Z_μ^- transversally at \tilde{p}_ϵ^- with $d(\tilde{p}_\epsilon^-, p_0^-) = \mathcal{O}(\epsilon)$ in A . Moreover, $W^s(\gamma_\epsilon^+)$ intersects $W^u(\tilde{\gamma}_\epsilon^-)$ and $W^u(\gamma_\epsilon^-)$ intersects $W^s(\tilde{\gamma}_\epsilon^+)$ in transverse heteroclinic orbits y_ϵ^1 and y_ϵ^2 , respectively.
- (iii) There exists $\delta_0 > 0$ and a trajectory $y_0(t, \bar{I}, \phi_0; \bar{\mu})$ of $(2.5)_0$ of the form (3.2) (with the necessary modifications) such that $y_0(t, \bar{I}, \phi_0; \bar{\mu})$ is $\mathcal{O}(\epsilon)$ -close to the orbits y_ϵ^1 and y_ϵ^2 outside of U^{δ_0} and
$$y_0(+\infty, \bar{I}, \phi_0; \bar{\mu}) = p_0^+, \quad y_0(-\infty, \bar{I}, \phi_0; \bar{\mu}) = p_0^-.$$
- (iv) If $\gamma_0^+ = \gamma_0^-$ in (A2) then $\gamma_\epsilon^+ = \tilde{\gamma}_\epsilon^+ = \gamma_\epsilon^- + \tilde{\gamma}_\epsilon^-$, and $y_\epsilon^1 = y_\epsilon^2$ is a transverse homoclinic orbit to γ_ϵ^+ , such that (iii) still holds.
- (v) (i)–(iv) are also satisfied for μ values sufficiently close to $\bar{\mu}$.

Proof: Most of the proof is virtually the same as that of theorem 5.5. The argument for transversality yields the expression

$$DG_\epsilon = \begin{pmatrix} D_x H_0 & 0 & 0 \\ 0 & D_I \Delta \mathcal{H} & D_\phi \Delta \mathcal{H} \\ 0 & D_I \mathcal{H}_\epsilon & D_\phi \mathcal{H}_\epsilon \end{pmatrix},$$

implying that for the matrix L_ϵ containing the first, third, and fourth column of DG_ϵ

$$\det(L_\epsilon)|_{q_0} = \epsilon D_{x_1} H_0|_{q_0} [\langle D \Delta \mathcal{H}, JD \mathcal{H} \rangle|_{p_0^+} + \mathcal{O}(\epsilon, \delta_0^2)],$$

holds with D denoting derivatives with respect to I and ϕ . This allows the same conclusion as in theorem 5.5. \square

We complete this section by noting that a corollary identical to corollary 5.6 could be stated in relation with theorem 5.7 too, but it is omitted for brevity.

6. Chaos associated with orbits homoclinic to resonances

We will discuss the ramifications of some of the phenomena listed in corollary 5.6. First we consider the case when the perturbed system $(2.1)_\epsilon$ has transverse homoclinic or heteroclinic orbits connecting periodic solutions and also study a certain bifurcation of transverse homoclinic orbits. Next we turn to the case of a homoclinic orbit connecting a saddle-saddle type fixed point in \mathcal{A}_ϵ to itself. Our methods do not apply immediately to the case of a saddle-center with a homoclinic orbit, which may arise in systems with an at least two-dimensional parameter μ , or with certain symmetries. For a general treatment of the saddle-center the reader is referred to Lerman [25], Mielke et al. [26], and the references cited therein. With a little bit of extra work (not detailed here) the energy-phase criterion applies to this case as well and gives results similar to the local results mentioned above on a larger domain.

6.1. Dynamics near orbits connecting periodic solutions in the resonance

Since this issue has been widely investigated and deeply understood, we simply invoke the Smale–Birkhoff homoclinic theorem (Smale [27]) to obtain the following

Theorem 6.1. Let us assume that hypotheses (H1) and (H2) or (H2') are satisfied and assumptions (A1)–(A3) hold. Let us further assume that $\gamma_0^+ = \gamma_0^-$ of theorem 5.5 or theorem 5.7 is a periodic orbit in A . Then certain iterates of appropriately defined Poincaré maps near $y_\varepsilon^1 = y_\varepsilon^2$ have Smale-horseshoes on the energy surface containing $y_\varepsilon^1 = y_\varepsilon^2$.

Theorem 6.1 implies that the Poincaré maps mentioned above possess invariant Cantor-sets on which they are homeomorphic to a full shift on N symbols. It follows from the structural stability of horseshoes that nearby energy surfaces also contain horseshoes in their dynamics. For more information on issues related to theorem 6.1 the reader is referred to Guckenheimer and Holmes [23] or Wiggins [19].

We remark that if γ_0^+ and γ_0^- are distinct periodic orbits of the reduced system then transverse *heteroclinic* connections between their perturbed counterparts can be destroyed by arbitrarily small further perturbation. However, since both of them are embedded in families of periodic orbits, and subsets of these families intersect Z_μ^+ and Z_μ^- , respectively, transversally, the family of heteroclinic connections between the two sets of periodic orbits is stable. If γ_0^+ and γ_0^- are identical periodic solutions, sufficiently small perturbation will preserve the *homoclinic* connection between them.

6.2. Saddle-node bifurcation of homoclinic orbits

Since the reduced system (4.10) is Hamiltonian, one periodic internal orbit indicates the existence of a one-parameter family of internal orbits. As we have seen earlier, internal orbits are energetically isolated, hence the periodic orbits in this family will have different energies. The majority of the orbits in such a family have transversal intersections with the zero sets Z_μ^+ and Z_μ^- (which of course includes the case of no intersection at all) but there might be an orbit, say, γ_0 in the family which is tangent to these sets. This suggests the presence of a periodic internal orbit γ_ε in \mathcal{A}_ε of the perturbed system $(2.1)_\varepsilon$, which energetically separates periodic internal orbits with no (or k) homoclinic connections from those with two (or $k + 2$) transversal homoclinic orbits. If we consider the energy as a bifurcation parameter, the appearance of two homoclinic orbits at the energy of γ_ε is reminiscent of the *saddle-node bifurcation* of a fixed point, thus we will use this term for it in our discussion.

Theorem 6.2. Let us assume that (H1)–(H2) hold and let γ_0 be a periodic orbit of the reduced system (4.10) contained in a one-parameter family of periodic internal orbits which are bounded in ϕ . Assume further that γ_0 is tangent to Z_μ^+ and Z_μ^- at the points $(0, \bar{\phi})$ and $(0, \bar{\phi} - \Delta\phi(\mu))$, respectively. Then, for $\varepsilon > 0$ small enough the perturbed system $(2.1)_\varepsilon$ has a periodic internal orbit γ_ε , $C^{r-1}\sqrt{\varepsilon}$ -close to γ_0 such that $W^s(\gamma_\varepsilon)$ and $W^u(\gamma_\varepsilon)$ have a *quadratic tangency* along a homoclinic orbit. Moreover, γ_ε divides \mathcal{A}_ε locally into two regions, one containing periodic solutions with a number $k \in \mathbb{Z}$ of transverse homoclinic orbits, the other containing periodic orbits with $k + 2$ transverse homoclinic orbits.

Proof: For simplicity assume that Z_μ^+ is connected (the proof will work for any of the connected components) and let $\mathcal{H}|_{\gamma_0} = h_0$. Then, as it was mentioned in the proof of theorem 5.5, for small ε Z_μ^+ perturbs into a set $Z_{\mu,\varepsilon}^+$ in \mathcal{A}_ε given by

$$\phi = g(\eta; \mu, \sqrt{\varepsilon}) = \bar{\phi} + \mathcal{O}(\varepsilon),$$

where g is a C^{r-1} function. Dividing (4.7) by $\sqrt{\varepsilon}$ we obtain that the level curves of the restricted

Hamiltonian satisfy the equation

$$\mathcal{H}(\eta, \phi; \mu) + \mathcal{O}(\sqrt{\epsilon}) = h, \quad (6.1)$$

with rescaled energy values h . Since $D_\phi \mathcal{H}(0, \bar{\phi}; \mu) \neq 0$, by the implicit function theorem (6.1) is equivalent to $\phi = G(\eta, h; \mu, \sqrt{\epsilon})$ in a neighborhood of $(0, \bar{\phi})$ where G is C^{r-2} in its arguments. Consider the equations

$$G(\eta, h; \mu, \sqrt{\epsilon}) - g(\eta; \mu, \sqrt{\epsilon}) = 0, \quad D_\eta G(\eta, h; \mu, \sqrt{\epsilon}) = 0, \quad (6.2)$$

which, by assumption, are solved by $(\eta, h, \sqrt{\epsilon}) = (0, h_0, 0)$. The implicit function theorem implies the existence of a smooth solution $(\eta(\mu, \sqrt{\epsilon}), h(\mu, \sqrt{\epsilon}))$ around $(0, h_0, 0)$ provided

$$D_h G D_\eta^2 G|_{(0, h_0; \mu, 0)} \neq 0 \quad (6.3)$$

holds, as it can be checked easily from (6.2). Substituting G for ϕ in (6.1) and differentiating with respect to h , then differentiating twice with respect to η yields

$$D_h G(0, h_0; \mu, 0) = \frac{1}{D_\phi \mathcal{H}(0, \bar{\phi}; \mu)}, \quad D_\eta^2 G(0, h_0; \mu, 0) = -\frac{m(I_r; \mu)}{D_\phi \mathcal{H}(0, \bar{\phi}; \mu)},$$

which together with hypothesis (H2) implies that (6.3) is satisfied for small ϵ ($D_\phi \mathcal{H}(0, \bar{\phi}; \mu) \neq 0$ since $D_\eta \mathcal{H}(0, \bar{\phi}; \mu) = 0$ and γ_0 is not an equilibrium). Hence we have obtained that the restricted system has a periodic internal orbit γ_ϵ , $C^{r-1} \sqrt{\epsilon}$ -close to γ_0 in A which has a quadratic tangency with $Z_{\mu, \epsilon}^+$ (see fig. 9).

Consider now ∂U^{δ_0} with $0 < \epsilon \ll \delta_0 \ll 1$, as in the proof of lemma 5.4. Let us define the set $W_{\delta_0}^s = W_{loc}^s(\mathcal{A}_\epsilon) \cap \partial U^{\delta_0}$ which is a two-dimensional manifold by the transversal intersection of the two manifolds in its definition. Moreover, using the fibering of $W_{loc}^s(\mathcal{A}_\epsilon)$ we conclude that $W_{\delta_0}^s$ is a C^{r-1} graph over an open subset of \mathcal{A}_ϵ . In other words, the fiber projection $\pi_s: \mathcal{F}_\epsilon^s \rightarrow \mathcal{A}_\epsilon$, $x \in f_\epsilon^s(p) \mapsto p$ is a C^{r-1} diffeomorphism restricted to $W_{\delta_0}^s$ (see section 4.2 for notation). It follows immediately that $C^s(\gamma_\epsilon) = W_{\delta_0}^s \cap W_{loc}^s(\gamma_\epsilon)$ is a smooth closed curve in $W_{\delta_0}^s$ since $C^s(\gamma_\epsilon) = \pi_s^{-1}(\gamma_\epsilon)$, as shown in fig. 10a.

Next we define the set $W_{\delta_0}^u = W^u(\mathcal{A}_\epsilon) \cap \partial U^{\delta_0}$. (Note that again, $W_{loc}^u(\mathcal{A}_\epsilon) \cap \partial U^{\delta_0} \subset W_{\delta_0}^u$ is a two-dimensional graph over \mathcal{A}_ϵ). Recall that solutions of (2.1) _{ϵ} contained in $W^u(\mathcal{A}_\epsilon)$ are $C^r \epsilon$ -close to solutions in $W^u(\mathcal{A}_0)$ outside U^{δ_0} . These latter solutions intersect ∂U^{δ_0} transversally when they approach

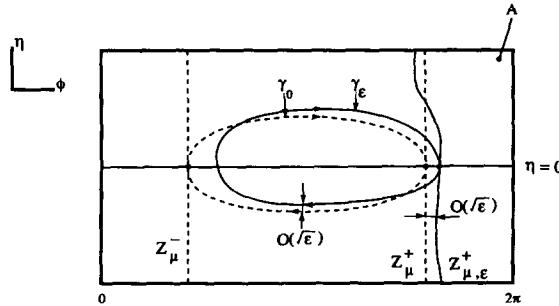


Fig. 9. The intersection of γ_ϵ and $Z_{\mu, \epsilon}^+$.

\mathcal{A}_ϵ again, thus $W_{\delta_0}^u$ has a component $W_{\delta_0}^{u^*}$ diffeomorphic to an open two-dimensional disk. Clearly, $W_{\delta_0}^{u^*}$ contains a closed smooth curve $C^u(\gamma_\epsilon) = W_{\delta_0}^{u^*} \cap W^u(\gamma_\epsilon)$, as seen in fig. 10b. If we denote the energy of the orbit γ_ϵ by $h_\epsilon = H|_{\gamma_\epsilon}$ and introduce the set $E_{\gamma_\epsilon} = E(h_\epsilon) \cap \partial U^{\delta_0}$ then obviously $C^s(\gamma_\epsilon), C^u(\gamma_\epsilon) \subset E_{\gamma_\epsilon}$ holds. Moreover, $\pi^{-1}(Z_{\mu,\epsilon}^+)$ will intersect E_{γ_ϵ} in exactly one point q_ϵ . Applying theorem 5.5, this means that $q_\epsilon = C^s(\gamma_\epsilon) \cap C^u(\gamma_\epsilon)$, as shown in fig. 10c. In the figure we used the fact that an open subset of E_{γ_ϵ} around $C^s(\gamma_\epsilon) \cup C^u(\gamma_\epsilon)$ is a two-dimensional graph over \mathcal{A}_ϵ , a property of $(2.1)_0$ which survives small perturbations.

What remains is to observe that since E_{γ_ϵ} is locally a graph over \mathcal{A}_ϵ , we can extend the fiber projection π_s from the compact subset $C^s(\gamma_\epsilon)$ to an open neighborhood \mathcal{K} of $C^s(\gamma_\epsilon)$ in E_{γ_ϵ} . If we denote this extension by Π_s then it is not hard to see that the image of \mathcal{K} under Π_s will look like as in fig. 11. Since $\Pi_s(C^u(\gamma_\epsilon) \cap \mathcal{K})$ and γ_ϵ lie on different sides of $Z_{\mu,\epsilon}^+$ in \mathcal{A}_ϵ , have a single common point, and γ_ϵ has a quadratic tangency with $Z_{\mu,\epsilon}^+$, we conclude that $\Pi_s(C^u(\gamma_\epsilon) \cap \mathcal{K})$ and γ_ϵ have a quadratic tangency in \mathcal{A}_ϵ . But Π_s is a C^{r-1} diffeomorphism ($r \geq 2$) thus $C^s(\gamma_\epsilon)$ and $C^u(\gamma_\epsilon)$ have a quadratic tangency in E_{γ_ϵ} . Now E_{γ_ϵ} is a local cross section within $E(h_\epsilon)$, which implies the first statement of the theorem.

Finally, we see that γ_ϵ is a Jordan curve in \mathcal{A}_ϵ , thus orbits inside γ_ϵ will not intersect $Z_{\mu,\epsilon}^+$. At the same time, by the smoothness of the family of periodic orbits, nearby orbits outside γ_ϵ will have two transversal intersections with $Z_{\mu,\epsilon}^+$. For these periodic internal orbits the proof of theorem 5.5 does not apply with ϵ fixed. However, our above construction of the extended fiber-projection Π_s does apply and proves the second statement of the theorem (see fig. 11 for the geometry of this construction). \square

Theorem 6.2 helps us to unveil the mechanism of the saddle-node bifurcation of homoclinic orbits. Figure 12 shows the creation of two transverse homoclinic orbits through the quadratic tangency of the

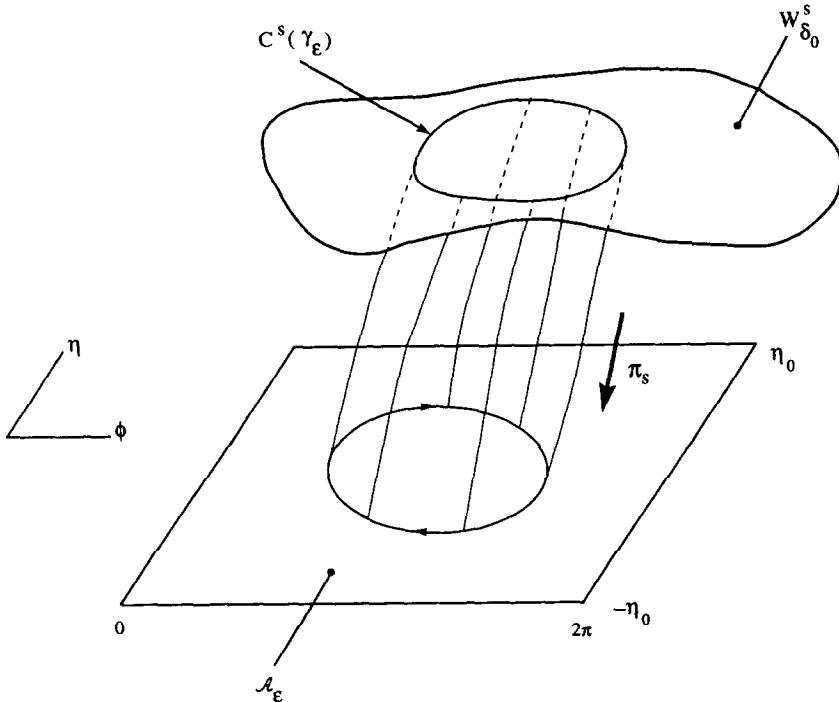


Fig. 10a. The proof of theorem 6.3.

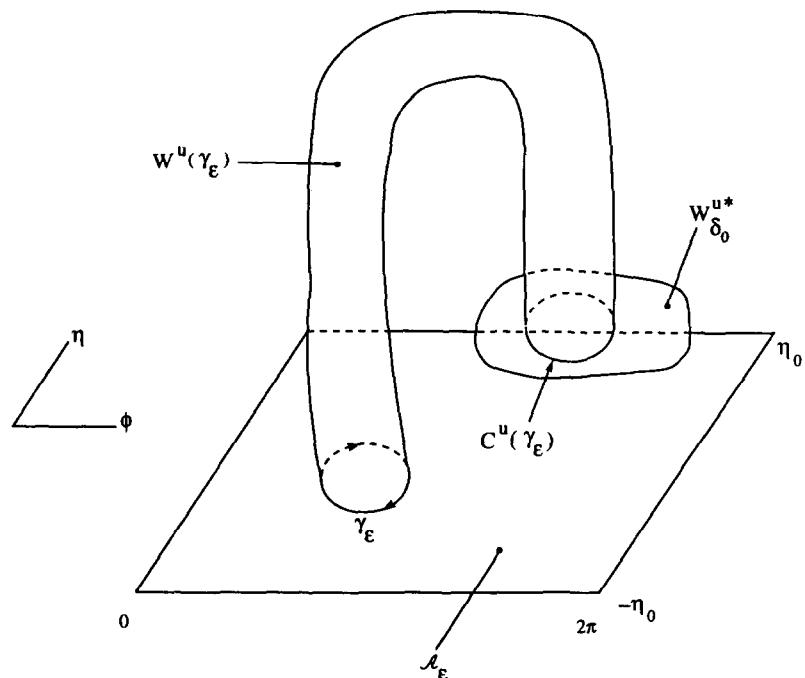


Fig. 10b. The proof of theorem 6.3 (cont.)

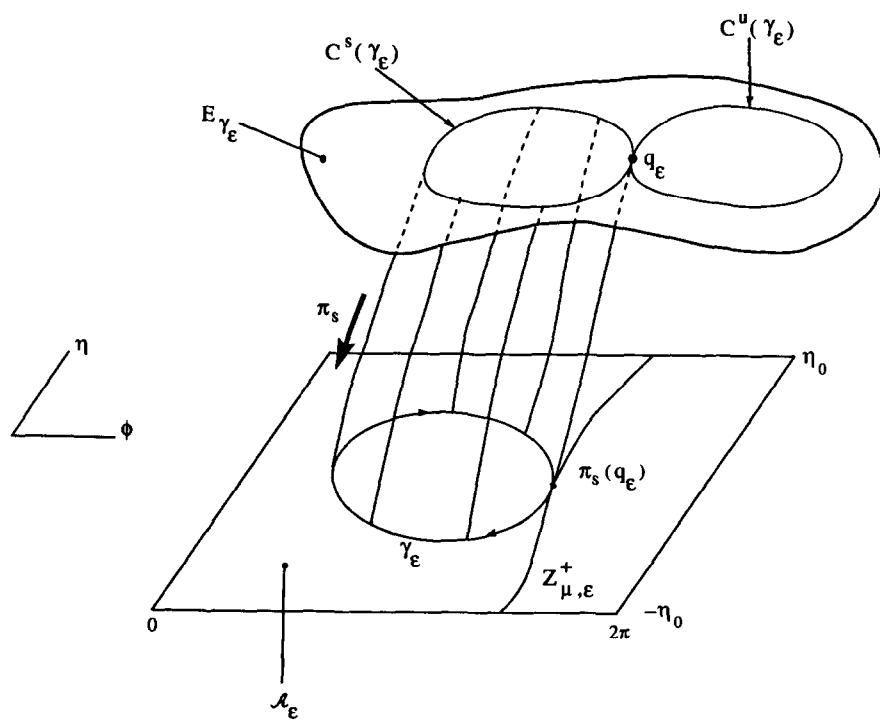


Fig. 10c. The proof of theorem 6.3 (cont.)

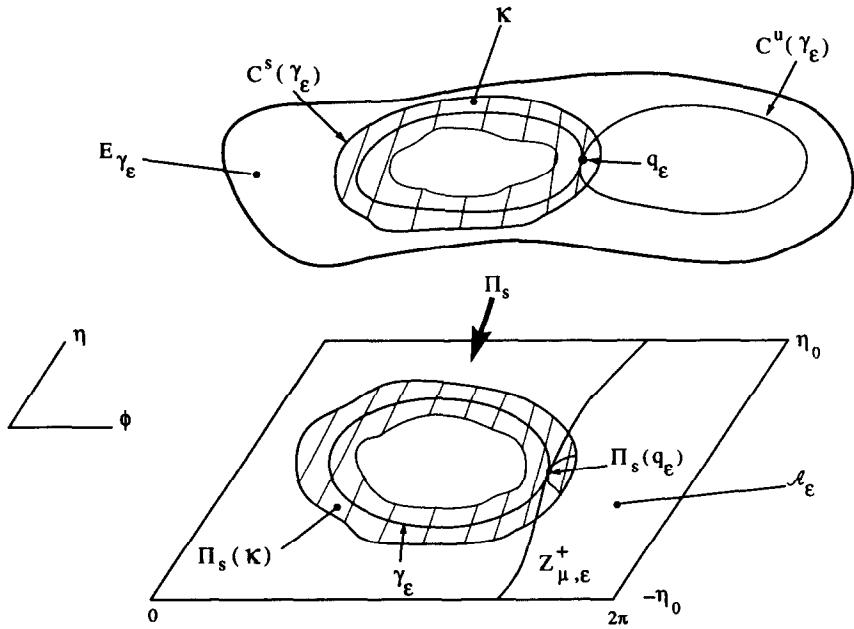


Fig. 11. The appearance of two transversal homoclinic orbits.

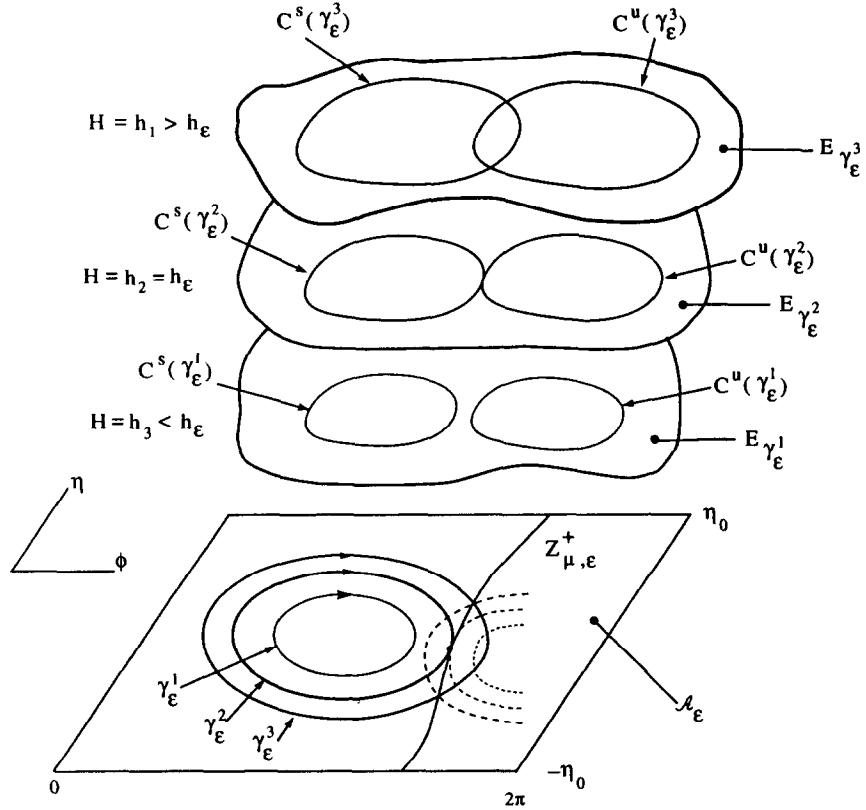


Fig. 12. The mechanism of the saddle-node bifurcation of homoclinic orbits.

two cylindrical manifolds $W^s(\mathcal{A}_\epsilon)$ and $W^u(\mathcal{A}_\epsilon)$. Here the bifurcation parameter is the energy h which is varied through a neighborhood of the bifurcation value h_ϵ . We assumed a particular situation in which the orbits contained inside γ_ϵ have lower energies than γ_ϵ . We also assumed that $Z_{\mu,\epsilon}^+$ has one connected component as in fig. 9. As we mentioned and indicated in theorem 6.2, if $Z_{\mu,\epsilon}^+$ has more than one connected component and we again vary the energy as a parameter then a tangency of the orbit corresponding to some value of h increases the number of transverse homoclinic orbits by two. An example of this repeated saddle-node bifurcation will be considered in section 7.

6.3. Dynamics near orbits homoclinic to a saddle-saddle in the resonance

We start by changing hypothesis (H1) slightly. From now on we will assume that for system $(2.1)_0$ the following holds:

(H1') There exist $I_1, I_2 \in U$, $I_1 < I_2$ such that for any $(I, \mu) \in [\tilde{I}_1, \tilde{I}_2] \times V$ $(2.1a)_0$ has a hyperbolic fixed point $\bar{x}_0(I_r; \mu)$ and a pair of homoclinic trajectories, $x^h(t, I_r; \mu)$ and $x^{h'}(t, I_r; \mu)$, which connect \bar{x}_0 to itself, as viewed in fig. 13a.

Remark 6.1. For future considerations, in fig. 13a we have assumed (without loss of generality) that $\bar{x}_0(I_r; \mu) = 0$, and the local stable and unstable manifolds of the origin are defined by $x_2 = 0$ and $x_1 = 0$, respectively. This assumption is just a convenient choice of coordinates and does not affect the results.

By adapting (H1'), we preserve the features of our unperturbed geometry in $(2.1)_0$ which were discussed in section 3, but we also add a new element to it: another homoclinic manifold Γ'_0 connecting \mathcal{A}_0 to itself, as shown in fig. 13b. If (H2) also holds we can repeat the analysis of the previous sections to detect some orbits of the standard form which perturb from orbits in Γ'_0 , and are homoclinic to the resonance in the sense as it was meant in section 5. We define the phase shift $\Delta\phi'(\mu)$ as in (3.3), but

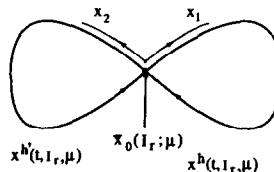


Fig. 13a. Hypothesis (H1').

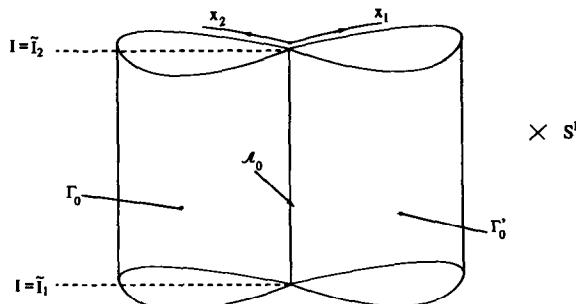


Fig. 13b. The geometry of the unperturbed standard form under (H1').

with $x^{h'}(t, I; \mu)$ instead of $x^h(t, I; \mu)$. Then we define the energy-difference function $\Delta\mathcal{H}'$ as in (5.5) using $\Delta\phi'(\mu)$ instead of $\Delta\phi(\mu)$, and the sets $Z_\mu'^+$ and $Z_\mu'^-$ as in (5.6) and (5.13), but with $\Delta\mathcal{H}'$ and $\Delta\phi'(\mu)$. We will assume that for the two phase-shifts

$$(H3) \quad \Delta\phi'(\mu) = -\Delta\phi(\mu)$$

holds which is satisfied, e.g. for a system studied in Haller and Wiggins [18]. By the symmetry assumed in (H3), we have $\Delta\mathcal{H}'(\phi; \mu) = \Delta\mathcal{H}(\phi - \Delta\phi(\mu); \mu)$, which implies $Z_\mu'^+ = Z_\mu^-$ and $Z_\mu'^- = Z_\mu^+$. It follows that whenever the energy-phase criterion of theorem 5.5 indicates an orbit homoclinic to the resonance in the perturbed system $(2.1)_\epsilon$ then another connection arising from the perturbation of Γ_0' exists on the “other side” of \mathcal{A}_ϵ . For concreteness, we assume that for $\mu \in (\mu_1, \mu_2)$ the reduced system (4.10) has a pair of internal orbits γ_0^a and γ_0^b , connecting the saddle points s_1 and s_2 in the annulus A with equal η coordinates and with ϕ coordinates ϕ_{s_1} and ϕ_{s_2} , respectively, satisfying $\phi_{s_2} = \phi_{s_1} + 2\pi$. By this latter condition on the angles, we will identify s_1 and s_2 in A and use the single notation s_0 for both. In fig. 14a we show the homoclinic internal orbits γ_0^a and γ_0^b together with the saddle point s_0 in A . We also assume that $Z_\mu'^+ = Z_\mu^-$ and $Z_\mu'^- = Z_\mu^+$ are located in A as in fig. 14a. More specifically, they intersect both the internal orbits γ_0^a and γ_0^b of the reduced system. We have also indicated two distances, d_u and d_s in fig. 14a for which we assume (cf. remark 6.2) that there exists $\mu^* \in (\mu_1, \mu_2)$ such that

$$(H4) \quad \lim_{\mu \rightarrow \mu^*} d_u = \lim_{\mu \rightarrow \mu^*} d_s = 0.$$

If we introduce local coordinates in the usual way such that the local stable and unstable manifolds of s_0 become coordinate axes, and d_u and d_s are small enough then they equal the distance of p_0^{a-} and p_0^{a+} , respectively, from the origin s_0 . The configuration depicted in fig. 14a is typical in potential-type reduced systems like (4.10).

Applying the energy-phase criterion (i.e. theorem 5.5 and corollary 5.6) to the situation drawn in fig. 14a, we obtain that for small $\epsilon > 0$ the perturbed system $(2.1)_\epsilon$ has two transverse homoclinic orbits, y_ϵ^a and y_ϵ^b as seen in fig. 14b, arising from the perturbation of Γ_0 and Γ_0' , respectively. The energy-phase criterion guarantees the existence of two more orbits but we do not deal with those for the moment. Note in fig. 14b that we have introduced local coordinates around the saddle point s_ϵ into which s_0 perturbs: (\tilde{x}_1, λ) are local coordinates on $W_{loc}^s(s_\epsilon)$, while (\tilde{x}_2, κ) parametrize $W_{loc}^u(s_\epsilon)$. The $(\tilde{x}_1, \tilde{x}_2)$ coordinate system is complementary to directions in \mathcal{A}_ϵ and is close to the coordinate system (x_1, x_2) shown in fig. 13a,b. The coordinates (κ, λ) are defined around s_ϵ in \mathcal{A}_ϵ , and locally $\kappa = 0$ on the one-dimensional stable manifold of s_ϵ in \mathcal{A}_ϵ (which contains the perturbed internal orbit γ_ϵ^a) and $\lambda = 0$ on the one-dimensional unstable manifold of s_ϵ in \mathcal{A}_ϵ (which contains the perturbed internal orbit γ_ϵ^b). To sum up, $W_{loc}^s(s_\epsilon)$ is defined by $(\tilde{x}_1, \kappa) = (0, 0)$ and $W_{loc}^u(s_\epsilon)$ is defined by $(\tilde{x}_2, \lambda) = (0, 0)$.

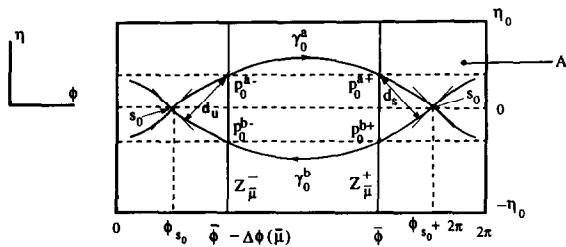


Fig. 14a. Assumption about the internal orbits γ_0^a and γ_0^b , and the sets Z_μ^+ and Z_μ^- .

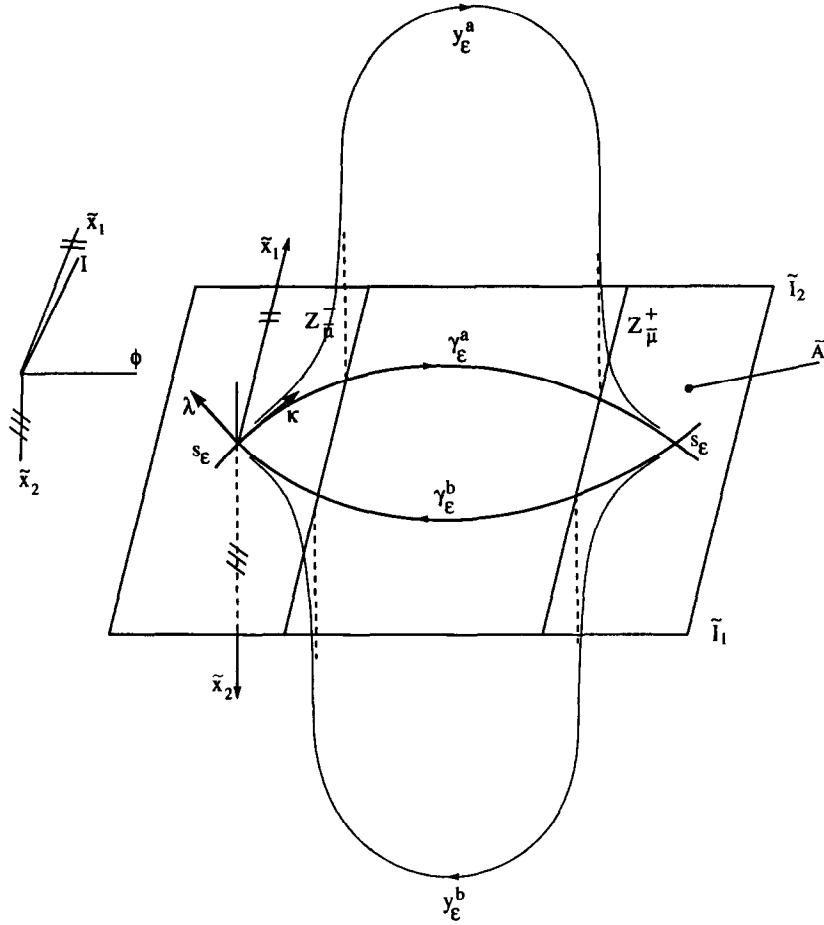
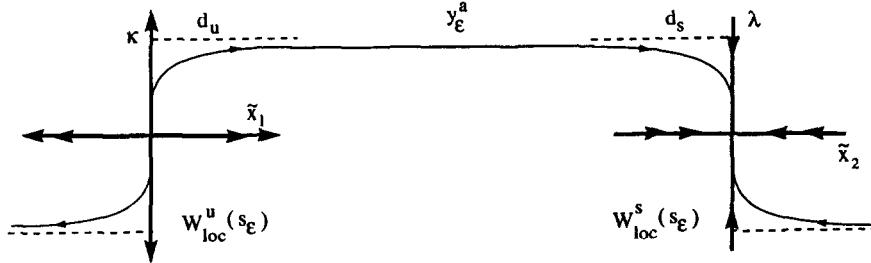


Fig. 14b. Transverse homoclinic orbits obtained from the energy-phase criterion.

Remark 6.2. The reason why we require (H4) to hold is that we want $p_0^{a\pm}$ and $p_0^{b\pm}$ to fall in the domain of definition of the local coordinate system described above in which the dynamics is topologically equivalent to the linearized dynamics at s_ϵ . This fact is used in a great deal regarding the “local map” around s_ϵ when one tries to construct a horseshoes on the energy surface $E(H|_{s_\epsilon})$ (cf. Holmes [28]).

The transverse homoclinic orbits y_ϵ^a and y_ϵ^b can be schematically represented in the coordinates discussed above as in fig. 15, where the distances d_u and d_s are those shown in fig. 14a. Here we used (iii) of theorem 5.5 which asserts that the transverse homoclinic connections stay close to unperturbed fibers outside of U^{δ_0} for small ϵ . The situation in fig. 15 was analyzed by Holmes [28] and – with more details – in Wiggins [19]. We will not invoke the construction of a chaotic invariant set near the saddle point, only verify the geometric conditions for chaos found in Holmes [28]. For details of the construction the reader may consult the two references mentioned above.

Following Holmes [28], we define the following manifolds:

Fig. 15. Schematic representation of y_ε^a and y_ε^b .

$$\begin{aligned}\Sigma^u &= \{(\tilde{x}_1, \tilde{x}_2, \kappa, \lambda) | |(\tilde{x}_1, \kappa)| = r, |(\tilde{x}_2, \lambda)| \leq \rho_u\}, \\ \Sigma^s &= \{(\tilde{x}_1, \tilde{x}_2, \kappa, \lambda) | |(\tilde{x}_1, \kappa)| \leq \rho_s, |(\tilde{x}_2, \lambda)| = r\},\end{aligned}\quad (6.4)$$

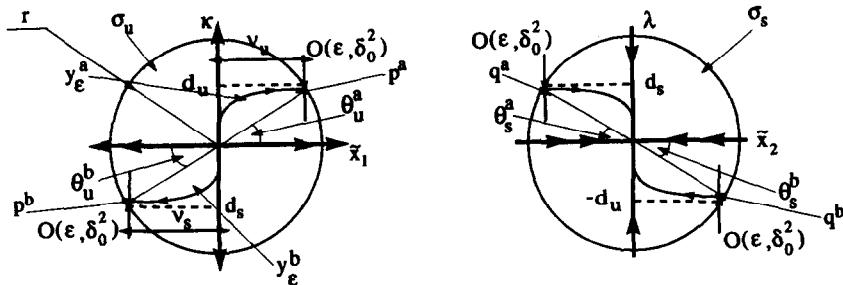
with positive constants r, ρ_s, ρ_u to be specified later. It is easy to see that the manifolds defined in (6.4) are solid 3-tori which are transversal to the flow of the perturbed system $(2.1)_\varepsilon$ for small positive ε . We also define

$$\begin{aligned}\Sigma_0^u &= \Sigma^u \cap E^\varepsilon(H|_{s_\varepsilon}), \quad \sigma^u = \Sigma_0^u \cap W_{loc}^u(s_\varepsilon), \\ \Sigma_0^s &= \Sigma^s \cap E^\varepsilon(H|_{s_\varepsilon}), \quad \sigma^s = \Sigma_0^s \cap W_{loc}^s(s_\varepsilon).\end{aligned}$$

Note that σ^s and σ^u are circles lying in the stable and unstable manifolds of s_ε , respectively. We introduce the notations

$$p^a = y_\varepsilon^a \cap \sigma^u, \quad p^b = y_\varepsilon^b \cap \sigma^u, \quad q^a = y_\varepsilon^a \cap \sigma^s, \quad q^b = y_\varepsilon^b \cap \sigma^s,$$

for the intersection points of the transverse homoclinic orbits with the circles σ^s and σ^u . In view of the newly defined geometric objects, we have redrawn fig. 15 in fig. 16 more carefully. As we see in the figure, the angles Θ_u and Θ_s , at which the two orbits enter and leave the domains bounded by σ^s and σ^u , can be approximated with an error of $\mathcal{O}(\varepsilon, \delta_0^2)$ by the angles corresponding to the straight lines $\kappa = d_u$ and $\lambda = d_s$. As it was used in the proof of theorem 5.5, these lines approximate unperturbed stable and unstable fibers with an error of $\mathcal{O}(\varepsilon, \delta_0^2)$, which have their basepoints in Z_μ^+ and Z_μ^- . It turns out from the analysis of Holmes [28] that the magnitude of the angles Θ_u and Θ_s is a crucial factor of the local dynamics around s_ε .

Fig. 16. Local geometry around s_ε . $\theta_u := \theta_u^a = \theta_u^b$ and $\theta_s := \theta_s^b = \theta_s^a$.

For fixed ε we linearize the perturbed flow at s_ε to obtain the eigenvalues $\pm l$ for the eigenspace tangent to the coordinate space $(\tilde{x}_1, \tilde{x}_2)$ and $\pm \sqrt{\varepsilon}k$ for the eigenspace tangent to \mathcal{A}_ε , with some $k, l > 0$. Since ε is small, $l > \sqrt{\varepsilon}k$, as it is assumed in Holmes [28]. We have now all the ingredients to state and prove.

Theorem 6.3. Let us assume that hypotheses (H1'), (H2)–(H4) and assumption (A1) of theorem 5.5 hold. Moreover, suppose that γ_0^a and γ_0^b are internal orbits of the reduced system (4.10) homoclinic to the saddle point s_0 , which intersect the sets Z_μ^+ and Z_μ^- (defined earlier) as shown in fig. 14a. Then, for sufficiently small $\varepsilon > 0$ there exists $\Delta\mu > 0$ such that if $|\mu - \mu^*| < \Delta\mu$ then:

- (i) An appropriately defined Poincaré map of system $(2.1)_\varepsilon$ has an invariant Cantor set on the energy surface $E^\varepsilon(H|_{s_\varepsilon})$. On the invariant set the Poincaré map is homeomorphic to a subshift of finite type on four symbols with (irreducible) transition matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

- (ii) Cantor sets homeomorphic to this invariant set also exist on energy surfaces close to $E^\varepsilon(H|_{s_\varepsilon})$.

Proof: First we set $0 < \varepsilon < \varepsilon_0$, where ε_0 is the same appearing in the statement of theorem 5.5. According to Holmes [28], to construct the Poincaré map above we have to satisfy the technical condition

$$\frac{\rho_u^l}{\rho_s^{\sqrt{\varepsilon}k}} > r^{(l-\sqrt{\varepsilon}k)}, \quad (6.5)$$

which affects the definition of Σ_0^u and Σ_0^s . We also select some sufficiently small $\Delta\mu > 0$ in accordance with remark 6.2. Assuming that (6.5) is satisfied, Holmes shows that (i) of our theorem 6.2 holds provided there exists a small nonzero positive number δ such that

$$|\tan \Theta_u \tan \Theta_s| < -\delta + \frac{l}{\sqrt{\varepsilon}k} \exp\left[\left(1 - \frac{l}{\sqrt{\varepsilon}k}\right) \ln \frac{\rho_s}{r}\right], \quad (6.6)$$

with Θ_u and Θ_s as defined in fig. 16, and for some suitable choice of r and ρ_s in (6.5). Let us fix some $r > d_u, d_s$ (see fig. 16). The distances ν_u and ν_s shown in fig. 16 can be computed as

$$\nu_s = \sqrt{r^2 - (r - d_s)^2}, \quad \nu_u = \sqrt{r^2 - (r - d_u)^2}. \quad (6.7)$$

Then, as discussed earlier, fig. 16 yields the estimates

$$\tan \Theta_u = \frac{d_u}{\nu_u} + \mathcal{O}(\varepsilon, \delta_0^2), \quad \tan \Theta_s = \frac{d_s}{\nu_s} + \mathcal{O}(\varepsilon, \delta_0^2),$$

which imply that

$$|\tan \theta_u \tan \Theta_s| = \frac{d_u d_s}{\nu_u \nu_s} + \mathcal{O}(\varepsilon, \delta_0^2).$$

We can now choose small positive ε_1 such that if $\varepsilon < \varepsilon_1$ then

$$|\tan \Theta_u \tan \Theta_s| < 2 \frac{d_u d_s}{\nu_u \nu_s}. \quad (6.8)$$

On the other hand, if we choose $\rho_s = r + \sqrt{\varepsilon}$ then using, say, the first equation of (6.7) we obtain

$$\frac{l}{\sqrt{\varepsilon} k} \exp \left[\left(1 - \frac{l}{\sqrt{\varepsilon} k} \right) \ln \frac{\rho_s}{r} \right] = \frac{1}{\sqrt{\varepsilon}} \frac{l}{k} \left(1 + \sqrt{\varepsilon} \frac{2d_u}{d_u^2 + \nu_u^2} \right)^{(1/\sqrt{\varepsilon})(l/k)-1}$$

Since we have

$$\lim_{\varepsilon \rightarrow 0} \left(1 + \sqrt{\varepsilon} \frac{2d_u}{d_u^2 + \nu_u^2} \right)^{(1/\sqrt{\varepsilon})(l/k)-1} = \exp \left(\frac{2d_u l}{k(d_u^2 + \nu_u^2)} \right),$$

we can choose positive ε_2 such that if $\varepsilon < \varepsilon_2$ then

$$\frac{l}{\sqrt{\varepsilon} k} \exp \left[\left(1 - \frac{l}{\sqrt{\varepsilon} k} \right) \ln \frac{\rho_s}{r} \right] > \frac{1}{\sqrt{\varepsilon}} \frac{l}{2k} \exp \left(\frac{2d_u l}{k(d_u^2 + \nu_u^2)} \right). \quad (6.9)$$

Clearly, there exists positive ε_3 such that if $\varepsilon < \varepsilon_3$ then

$$2 \frac{d_u d_s}{\nu_u \nu_s} < \frac{1}{\sqrt{\varepsilon}} \frac{l}{2k} \exp \left(\frac{2d_u l}{k(d_u^2 + \nu_u^2)} \right). \quad (6.10)$$

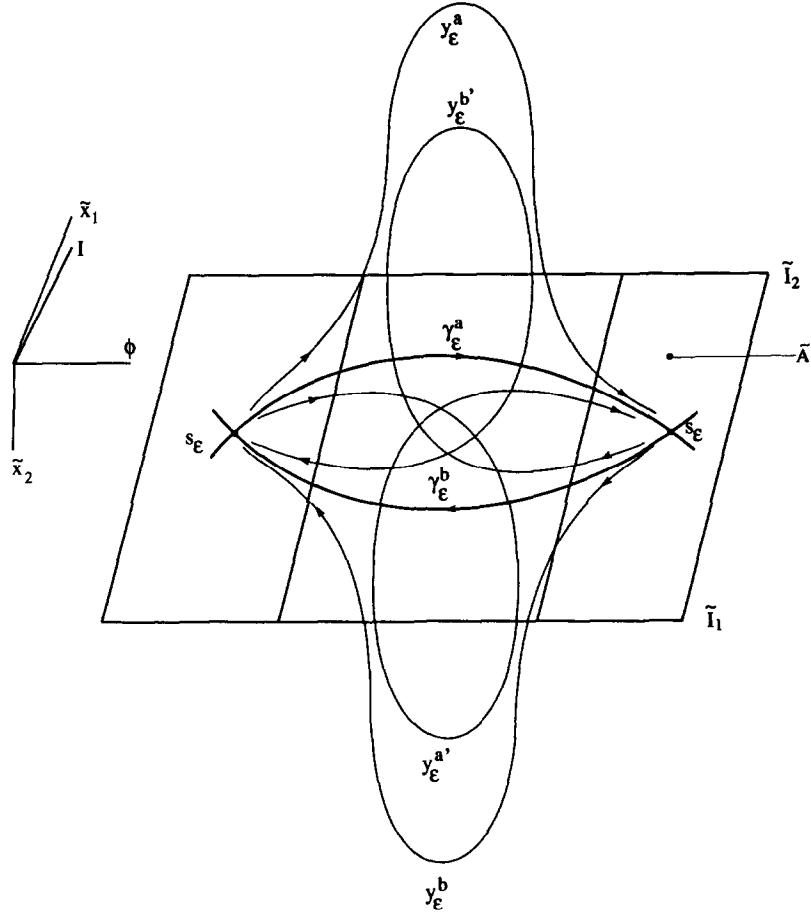
But (6.8)–(6.10) show that selecting $\varepsilon < \min(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ sufficiently small we can find $\delta > 0$ verifying (6.6). We choose $\rho_u > 2^{\sqrt{\varepsilon} k/l} r$ to satisfy (6.5), which completes the proof of (i). Finally, the structural stability of horseshoes and (v) of theorem 5.5 proves (ii). \square

We do not repeat the results for the case when hypothesis (H2') holds, only point out that it is again possible. The difference compared to theorem 6.3 is the fact that the sets Z_μ^+ and Z_μ^- do not necessarily look like as those in fig. 14a, which would make the formulation of the assumptions for a theorem analogous to theorem 6.3 slightly more complicated.

We close this section by noting that the set-up of fig. 14a implies the presence of another pair of transverse homoclinic orbits $y_\varepsilon^{a'}$ and $y_\varepsilon^{b'}$, as indicated in fig. 17. However, the results of Holmes [28] are not applicable to study the interaction of these two orbits. We found that other approaches to construct horseshoes also fail for these orbits, as well as for the previously discussed orbits in case hypothesis (H3) is not satisfied. It appears that the energy level ‘‘bounding’’ the resonance does not admit horseshoes in system (2.1) _{ε} if (H3) and (H4) do not hold. This may be related to the fact that the topology of the horseshoes below and above these energy levels are different and the transversality of these invariants sets is momentarily lost at the critical values of the energy.

7. An example

In this section we apply our theory to an example. The system we study arises from a two-Fourier-mode truncation of the damped and driven nonlinear Schrödinger equation that is described in Bishop

Fig. 17. The four homoclinic orbits y_ε^a , y_ε^b , $y_\varepsilon^{a'}$ and $y_\varepsilon^{b'}$.

et al. [16,17]. The theory of Kovačič and Wiggins [15] was used to study this problem in the dissipative case. We study the same system, but with zero dissipation. For the details of the derivation of the two-mode model from the nonlinear Schrödinger equation we refer the reader to Bishop et al. [16].

The equations that we will study are given by

$$\begin{aligned} \dot{x}_1 &= -k^2 x_2 - \frac{3}{4} x_1^2 x_2 + \frac{1}{4} x_2^3 - \varepsilon \left(\Gamma \frac{x_2}{\sqrt{2I - x_1^2 - x_2^2}} \sin \phi \right), \\ \dot{x}_2 &= (k^2 - 2I)x_1 + \frac{7}{4}x_1^3 + \frac{3}{4}x_1x_2^2 + \varepsilon \left(\Gamma \frac{x_1}{\sqrt{2I - x_1^2 - x_2^2}} \sin \phi \right), \\ \dot{I} &= -\varepsilon \left(\Gamma \sqrt{2I - x_1^2 - x_2^2} \cos \phi \right), \\ \dot{\phi} &= I - 1 + x_1^2 - \varepsilon \Gamma \frac{1}{\sqrt{2I - x_1^2 - x_2^2}} \sin \phi, \end{aligned} \quad (7.1)_\varepsilon$$

which are of the following form:

$$\dot{x}_1 = \frac{\partial H_0}{\partial x_2} + \varepsilon \frac{\partial H_1}{\partial x_2}, \quad \dot{x}_2 = -\frac{\partial H_0}{\partial x_1} - \varepsilon \frac{\partial H_1}{\partial x_1},$$

$$\dot{I} = -\varepsilon \frac{\partial H_1}{\partial \phi}, \quad \dot{\phi} = \frac{\partial H_0}{\partial I} + \varepsilon \frac{\partial H_1}{\partial I},$$

where

$$H_0 = \frac{1}{2}I^2 - I - \frac{7}{16}x_1^4 - \frac{3}{8}x_1^2x_2^2 + \frac{1}{16}x_2^4 + (I - \frac{1}{2}k^2)x_1^2 - \frac{1}{2}k^2x_2^2,$$

and

$$H_1 = \Gamma \sqrt{2I - x_1^2 - x_2^2} \sin \phi.$$

The reader can easily see that these equations have the general form of $(2.1)_\varepsilon$. (Note: in making the correspondence between $(7.1)_\varepsilon$ and the equations studied by Kovačić and Wiggins [15] the substitution $(x_1, x_2, I, \phi) \rightarrow (x, y, I, -\gamma)$ must be made.) Note that there are two parameters in $(7.1)_\varepsilon$ – k and Γ . Hence, in our notation for the general theory $\mu = (k, \Gamma)$.

7.1. The geometric structure of the unperturbed system

The unperturbed system is given by

$$\begin{aligned} \dot{x}_1 &= -k^2x_2 - \frac{3}{4}x_1^2x_2 + \frac{1}{4}x_2^3 = \frac{\partial H_0}{\partial x_2}, \\ \dot{x}_2 &= (k^2 - 2I)x_1 + \frac{7}{4}x_1^3 + \frac{3}{4}x_1x_2^2 = -\frac{\partial H_0}{\partial x_1}, \\ \dot{I} &= 0 = -\frac{\partial H_0}{\partial \phi}, \quad \dot{\phi} = I - 1 + x_1^2 = \frac{\partial H_0}{\partial I}. \end{aligned} \tag{7.1}_0$$

It can easily be shown that the x_1 - x_2 component of $(7.1)_0$ has a hyperbolic fixed point at $(x_1, x_2) = (0, 0)$ for all $I > \frac{1}{2}k^2$, which is connected to itself by a pair of homoclinic orbits, thus system $(7.1)_0$ satisfies our hypothesis (H1) (it also satisfies (H1')). Furthermore,

$$\mathcal{A}_0 = \{(x_1, x_2, I, \phi) | x_1 = x_2 = 0, I > \frac{1}{2}k^2\} \tag{7.2}$$

is a two-dimensional, normally hyperbolic invariant manifold, connected to itself by a symmetric pair of three-dimensional homoclinic manifolds (described more fully in Kovačić [24] and Kovačić and Wiggins [15]). Introducing

$$\Gamma_0 = W^s(\mathcal{A}_0) \cap W^u(\mathcal{A}_0) = \{(x_1, x_2, I, \phi) | H_0(x_1, x_2, I) - H_0(0, 0, I) = 0, x \neq 0\},$$

we have a homoclinic structure like that shown in fig. 13b.

The dynamics on \mathcal{A}_0

Using (7.1)₀ and (7.2), the vector field restricted to \mathcal{A}_0 is easily seen to be

$$\dot{I} = 0, \quad \dot{\phi} = I - 1.$$

We see that a resonance occurs at $I = 1$ giving rise to a circle of fixed points on \mathcal{A}_0 , thus hypothesis (H2) of section 2 is satisfied.

The phase shifts

In Kovacić and Wiggins [15] the phase shift $\Delta\phi$ is calculated for orbits heteroclinic to points on the circle of fixed points. There are two cases, depending on the value of k :

$$0 < k < \frac{1}{2}:$$

$$\Delta\phi(k) = 2 \cot^{-1} \left(\frac{k}{\sqrt{2-k^2}} \right) + \frac{2}{\sqrt{7}} \tanh^{-1} \left(\sqrt{\frac{7k^2}{2-k^2}} \right) - \pi;$$

$$\frac{1}{2} < k < \sqrt{2}:$$

$$\Delta\phi(k) = 2 \tan^{-1} \left(\frac{\sqrt{2-k^2}}{k} \right) + \frac{2}{\sqrt{7}} \tanh^{-1} \left(\sqrt{\frac{2-k^2}{7k^2}} \right).$$

The phase shift, as a function of k , is graphed in fig. 18.

7.2. The geometric structure of the perturbed system

As a result of the symmetry $(x_1, x_2, I, \phi) \rightarrow (-x_1, -x_2, I, \phi)$, \mathcal{A}_e and \mathcal{A}_0 are identical (in other words, $(x_1, x_2) = (0, 0)$ is a fixed point of the x_1 - x_2 components of the equation for both the unperturbed and perturbed systems). This greatly simplifies some of our analysis, in particular, the computation of the perturbed system restricted to \mathcal{A}_e .

The dynamics on \mathcal{A}_e

Using equations (2.2), (4.8), and the fact that $\bar{x}_0(I_r; \mu) = (0, 0)$, we find the Hamiltonian of the reduced system of the form

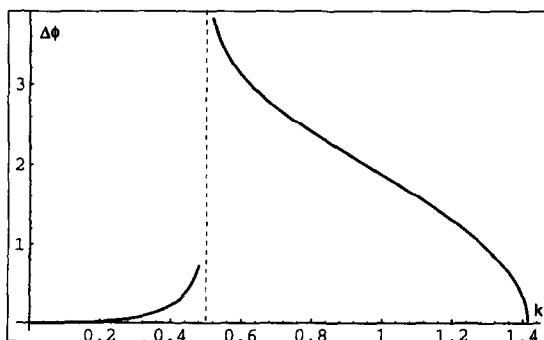


Fig. 18. The graph of $\Delta\phi$.

$$\mathcal{H}(\eta, \phi; \mu) = \frac{1}{2}\eta^2 + \Gamma\sqrt{2}\sin\phi.$$

Hence, from (4.10) the *reduced system* is given by

$$\eta' = -\Gamma\sqrt{2}\cos\phi, \quad \phi' = \eta. \quad (7.3)$$

Notice that (7.3) is just the equation for the familiar simple pendulum. This system has a hyperbolic fixed point at

$$q_0 = (0, \frac{1}{2}\pi),$$

and an elliptic fixed point at

$$p_0 = (0, \frac{3}{2}\pi).$$

The hyperbolic fixed point is connected to itself by a pair of homoclinic orbits (see fig. 19). We now fix some $\eta_0 > 2\sqrt{\sqrt{2}\Gamma}$, so that all the rotational orbits and separatrices of the pendulum are internal orbits, in accordance with our discussion at the end of section 5.3.

7.3. The energy-phase criterion and the existence of homoclinic orbits

We are now in a position to apply the theory of section 5 and determine the existence of orbits homoclinic to the periodic orbits inside the resonance as well as orbits homoclinic to the saddle fixed point. This amounts to applying the energy-phase criterion as formulated in theorem 5.5. A simple calculation gives

$$\Delta\mathcal{H}(\phi; \Gamma, k) = \mathcal{H}(\eta, \phi; \Gamma) - \mathcal{H}(\eta, \phi - \Delta\phi(k); \Gamma) = 0 \Rightarrow \sin\phi - \sin(\phi - \Delta\phi) = 0. \quad (7.4)$$

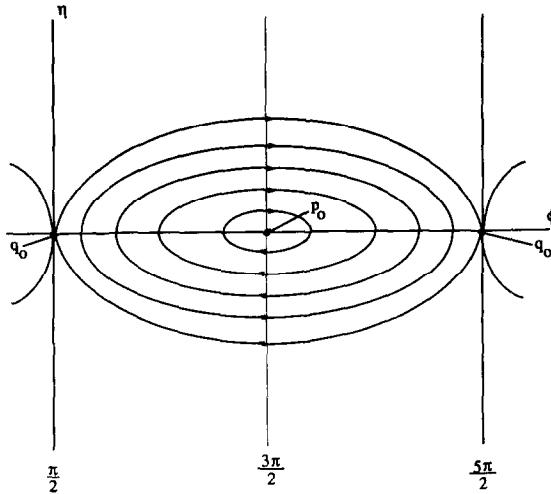


Fig. 19. The phase portrait of the reduced system.

It is easy to see that (7.4) has two solutions given by

$$\phi_1 = \frac{1}{2}\Delta\phi + \frac{1}{2}\pi \quad (7.5)$$

and

$$\phi_2 = \frac{1}{2}\Delta\phi + \frac{3}{2}\pi. \quad (7.6)$$

Using (7.12), we can show that for $\Delta\phi(k) = 0, 2\pi$ we have $D_\phi \Delta\mathcal{H}(\phi; \Gamma, k) = 0$ at $\phi_1 = \frac{1}{2}\pi, \frac{3}{2}\pi$ and $\phi_2 = \frac{1}{2}\pi, \frac{3}{2}\pi$, i.e. $D_\phi \Delta\mathcal{H}$ is zero at the ϕ values of the hyperbolic and elliptic fixed points in the resonance. According to fig. 18 this means that for $\Delta\phi \in [0, 2\pi]$ there are two values of k for which $\Delta\mathcal{H}(\phi; \Gamma, k) = 0$ and $D_\phi \Delta\mathcal{H}(\phi; \Gamma, k) = 0$. Using the ϕ periodicity of $\Delta\mathcal{H}(\phi; \Gamma, k)$, it follows that there is a set Λ of countable infinity of k values (converging to $k = \frac{1}{2}$ from above and below) for which $D_\phi \Delta\mathcal{H}(\phi; \Gamma, k) = 0$.

Using (7.5) and (7.6) we have

$$Z_{\mu,1}^+ = \{(\eta, \phi) | \phi = \frac{1}{2}\Delta\phi(k) + \frac{1}{2}\pi\}, \quad Z_{\mu,1}^- = \{(\eta, \phi) | \phi = \frac{1}{2}\pi - \frac{1}{2}\Delta\phi(k)\},$$

$$Z_{\mu,2}^+ = \{(\eta, \phi) | \phi = \frac{1}{2}\Delta\phi(k) + \frac{3}{2}\pi\}, \quad Z_{\mu,2}^- = \{(\eta, \phi) | \phi = \frac{3}{2}\pi - \frac{1}{2}\Delta\phi(k)\},$$

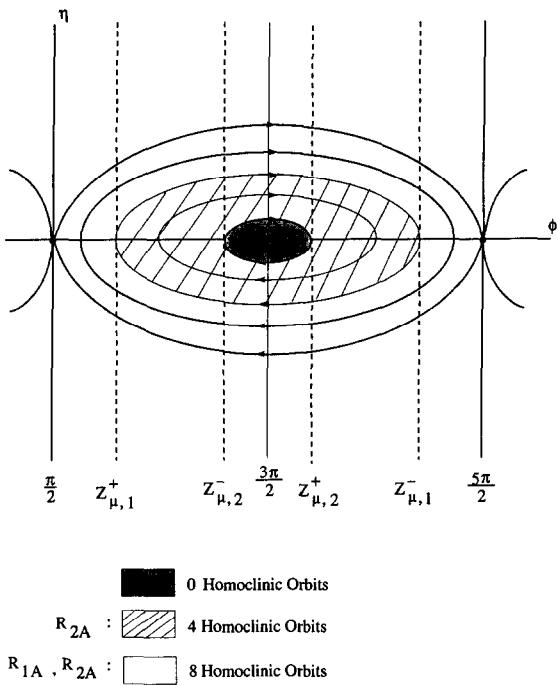
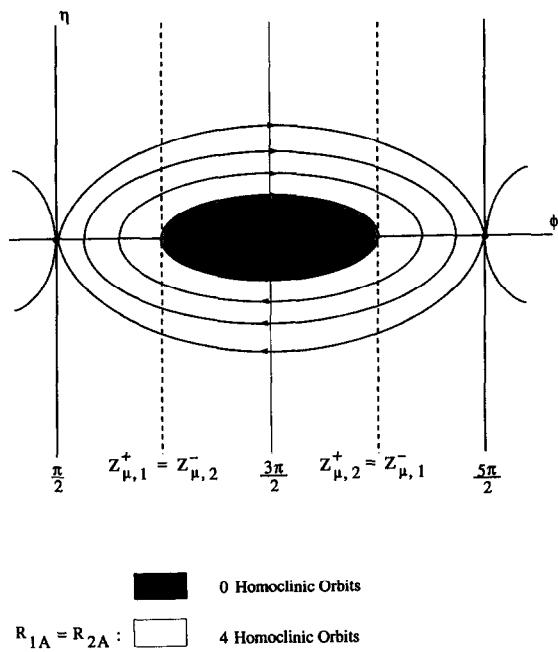
defined for $k \notin \Lambda$.

For $\Delta\phi = 0 \bmod 2\pi$ ($k \in \Lambda$), $Z_{\mu,1}^+$ and $Z_{\mu,1}^-$ are not defined but would be located at $\phi = \frac{1}{2}\pi, \frac{5}{2}\pi$. As $\Delta\phi$ increases, $Z_{\mu,1}^+$ moves from $\phi = \frac{1}{2}\pi$ monotonically towards $\frac{3}{2}\pi$ (with ϕ increasing) and $Z_{\mu,1}^-$ moves monotonically towards $\frac{3}{2}\pi$ (with ϕ decreasing), reaching these points at $\Delta\phi = 2\pi$, where they again cease to be defined.

Similarly, for $\Delta\phi = 0 \bmod 2\pi$ ($k \in \Lambda$) $Z_{\mu,2}^+$ and $Z_{\mu,2}^-$ are not defined but would lie at $\frac{3}{2}\pi$. As $\Delta\phi$ increases, $Z_{\mu,2}^+$ moves monotonically from $\phi = \frac{3}{2}\pi$ towards $\frac{5}{2}\pi$ (with ϕ increasing) and $Z_{\mu,2}^-$ moves monotonically towards $\frac{1}{2}\pi$ (with ϕ decreasing), reaching these points at $\Delta\phi = 2\pi$, where they cease to be defined. Also, at $\Delta\phi = \pi$ $Z_{\mu,2}^+$ coincides with $Z_{\mu,1}^-$ and $Z_{\mu,1}^+$ coincides with $Z_{\mu,2}^-$. This variation of the sets $Z_{\mu,1}^\pm$ and $Z_{\mu,2}^\pm$ repeats itself more and more rapidly as the parameter k approaches $\frac{1}{2}$ (see fig. 18).

In figs. 20a–c we plot the location of $Z_{\mu,1}^+$, $Z_{\mu,1}^-$, $Z_{\mu,2}^+$, and $Z_{\mu,2}^-$ for a value of $\Delta\phi$ between 0 and π , for $\Delta\phi = \pi$, and for a value of $\Delta\phi$ between π and 2π , respectively. These three plots also indicate the main qualitatively different behaviors possible. We discuss each situation individually, but first, we want to introduce some terminology. As vertical lines, $Z_{\mu,1}^+$ and $Z_{\mu,1}^-$ are tangent to the right and left, respectively, extremal (in ϕ) points of a unique periodic orbit. All periodic orbits surrounding this orbit are intersected by $Z_{\mu,1}^+$ and $Z_{\mu,1}^-$ in two unique points, respectively (the homoclinic orbits are intersected in one point). The periodic orbits inside this periodic orbit do not intersect $Z_{\mu,1}^+$ and $Z_{\mu,1}^-$. We refer to the region outside this special periodic orbit as the *1-accessible region*, denoted R_{1A} . Similarly, a *2-accessible region* can be defined using $Z_{\mu,2}^+$ and $Z_{\mu,2}^-$ and will be denoted R_{2A} .

In figs. 20a–c we indicate the 1-accessible and 2-accessible regions. From our earlier discussions, periodic orbits that are in both R_{1A} and R_{2A} have eight transverse homoclinic orbits, and periodic orbits that are neither in R_{1A} nor R_{2A} have no (simple) transverse homoclinic orbits. (The reason the numbers are 0–4–8, rather than 0–2–4, is because the symmetry of the system effectively doubles the number of transverse homoclinic orbits.) The saddle point q_0 has 8 transverse homoclinic orbits. Thus, for $0 < \Delta\phi < \pi$ and $\pi < \Delta\phi < 2\pi$ the interior of the resonance (i.e., the interior of the region bounded by

Fig. 20a. The energy-phase criterion for $\Delta\phi \in (0, \pi)$.Fig. 20b. The energy-phase criterion for $\Delta\phi = \pi$.

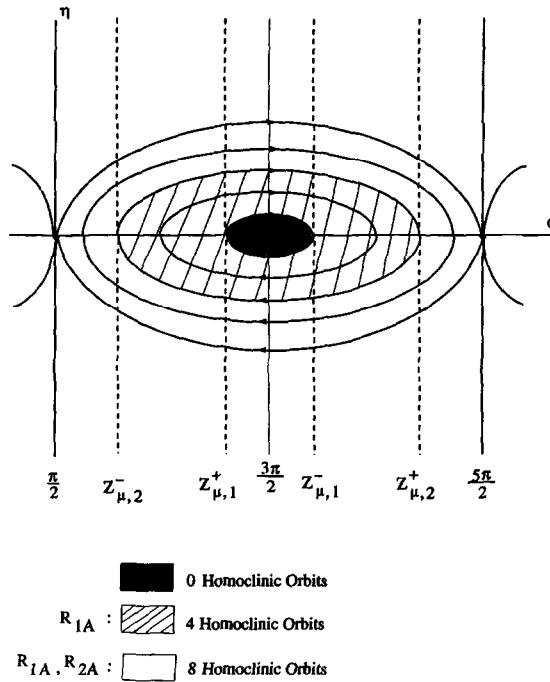


Fig. 20c. The energy-phase criterion for $\Delta\phi \in (\pi, 2\pi)$.

the transverse homoclinic orbits connecting q_0) is partitioned into three regions; one containing periodic orbits that have no transverse homoclinic orbits, one containing periodic orbits that have four transverse homoclinic orbits, and one containing periodic orbits that have eight transverse homoclinic orbits. In passing between these regions a *saddle-node bifurcation of homoclinic orbits* occurs as described in section 6. Concerning chaos, associated with each transverse homoclinic orbit to a hyperbolic periodic orbit is a Smale-horseshoe with its attendant chaotic dynamics (cf. theorem 6.1). Unfortunately, the orbits homoclinic to the saddle point *do not* satisfy hypothesis (H4) of section 6.2, hence theorem 6.3 does not apply.

Acknowledgements

This research has been supported by an NSF Presidential Young Investigator Award and an ONR Young Investigator Award.

Note added in proof

After this manuscript was submitted Gregor Kovačič announced some results in ref. [29] for periodic internal orbits, covering (i) of our corollary 5.6 and theorem 6.1. His proof for transversality makes extensive use of the second assumption of our hypothesis (H2) and does not generalize to the case of the normally hyperbolic 2-manifold of equilibria in our hypothesis (H2').

References

- [1] V.I. Arnold, V.V. Kozlov and A.I. Neishtadt, Mathematical aspects of classical and celestial mechanics, in: *Dynamical Systems III*, V.I. Arnold, ed. (Springer, New York, 1988).
- [2] J. Pöschel, On elliptic lower dimensional tori in Hamiltonian systems, *Math. Z.* 202 (1989) 559.
- [3] L.H. Eliasson, Perturbations of stable invariant tori, *Ann. Sci. Norm. Super. Pisa Cl. Sci. IV. Ser.* 15 (1988) 115.
- [4] D.V. Treshchev, The mechanism of destruction of resonant tori of Hamiltonian systems, *Math. USSR Sb.* 68 (1991) 181.
- [5] R. de la Llave and C.E. Wayne, Whiskered and low dimensional tori in nearly integrable Hamiltonian systems, University of Texas, Austin preprint (1990).
- [6] S.M. Graff, On the conservation of hyperbolic invariant tori for Hamiltonian systems, *J. Diff. Eq.* 15 (1974) 1.
- [7] V.I. Arnold, Instability of dynamical systems with many degrees of freedom, *Dokl. Akad. Nauk. USSR* 156 (1964) 9.
- [8] P. Holmes and J. Marsden, Horseshoes in perturbations of Hamiltonian systems with two degrees of freedom, *Commun. Math. Phys.* 82 (1982) 523.
- [9] V.K. Melnikov, On the stability of the center for time periodic perturbations, *Trans. Mosc. Math. Soc.* 12 (1963) 1.
- [10] Z.C. Feng and P.R. Sethna, Global bifurcations in the motion of parametrically excited thin plates, University of Minnesota preprint (1992).
- [11] Z.C. Feng and P.R. Sethna, Symmetry breaking bifurcations in resonant surface waves, *J. Fluid Mech.* 199 (1989) 495.
- [12] Z.C. Feng and P.R. Sethna, Global bifurcation and chaos in parametrically forced systems with one-one resonance, *Dyn. Stab. Syst.* 5 (1990) 201.
- [13] Z.C. Feng and S. Wiggins, On the existence of chaos in a class of two-degrees-of-freedom, damped, strongly parametrically forced mechanical systems with broken $O(2)$ symmetry, *Z. Angew. Math. Phys.*, to appear.
- [14] X.L. Yang and P.R. Sethna, Non-linear phenomena in forced vibrations of a nearly square plate: antisymmetric case, *J. Sound Vib.* 155 (1992) 413.
- [15] G. Kovačič and S. Wiggins, Orbits homoclinic to resonances, with an application to chaos in the damped and forced sine-Gordon equation, *Physica D* 57 (1992) 185.
- [16] A.R. Bishop, M.G. Forest, D.W. McLaughlin and E.A. Overman, A modal representation of chaotic attractors for the driven, damped pendulum chain, *Phys. Lett. A* 1444 (1990) 17.
- [17] A.R. Bishop, R. Flesch, M.G. Forest, D.W. McLaughlin and E.A. Overman, Correlations between chaos in a perturbed sine-Gordon equation and a truncated model system, *SIAM J. Math. Anal.* 21 (1990) 1511.
- [18] G. Haller and S. Wiggins, Geometry and chaos near resonant equilibria of 3DOF Hamiltonian systems, Caltech preprint (1992).
- [19] S. Wiggins, *Global Bifurcations and Chaos – Analytical Methods* (Springer, New York, 1988).
- [20] N. Fenichel, Persistence and smoothness of invariant manifolds for flows, *Ind. Univ. Math. J.* 21 (1971) 193.
- [21] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, *J. Diff. Eqs.* 31 (1979) 53.
- [22] L.M. Lerman and Ya.L. Umanskii, Structure of the Poisson action of R^2 on a four dimensional symplectic manifold I., *Sel. Math. Sov.* 6 (1987) 365.
- [23] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vectorfields* (Springer, New York, 1983).
- [24] G. Kovačič, Chaos in a model of the forced and damped sine-Gordon equation, PhD. thesis, Caltech (1989).
- [25] L.M. Lerman, Hamiltonian systems with a loop of a separatrix of a saddle-center, *Sel. Math. Sov.* 10 (1990) 297.
- [26] A. Mielke, P. Holmes, and O. O'Reilly, Cascades of homoclinic orbits to, and chaos near, a Hamiltonian saddle-center, *J. Dyn. Diff. Eqs.* 4 (1992) 95.
- [27] S. Smale, Diffeomorphisms with many periodic points, in: *Differential and Combinatorial Topology*, S.S. Cairns, ed. (Princeton Univ. Press, Princeton, 1963) p. 63.
- [28] P. Holmes, Periodic, nonperiodic and irregular motions in a Hamiltonian system, *Rocky Mountain J. Math.* 10 (1980) 679.
- [29] G. Kovačič, Singular perturbation theory for homoclinic orbits in a class of near-integrable Hamiltonian systems, RPI preprint (1992).