Instabilities in the dynamics of neutrally buoyant particles

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The asymptotic dynamics of finite-size particles is governed by a slow manifold that is globally attracting for sufficiently small Stokes numbers. For neutrally buoyant particles (suspensions), the slow dynamics coincide with that of infinitesimally small particles, therefore the suspension dynamics should synchronize with Lagrangian particle motions. Paradoxically, recent studies observe a scattering of suspension dynamics along Lagrangian particle motions. Here we resolve this paradox by proving that despite its global attractivity, the slow manifold has domains that repel nearby passing trajectories. We derive an explicit analytic expression for these unstable domains; we also obtain a necessary condition for the global attractivity of the slow manifold. We illustrate our results on neutrally buoyant particle motion in a two-dimensional model of vortex shedding behind a cylinder in crossflow and on the three-dimensional steady Arnold–Beltrami–Childress flow.
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I. INTRODUCTION

It is well known that the velocity of a finite-size spherical particle typically differs from the local velocity vector of the ambient fluid flow (cf. Maxey and Riley[1]). For small enough particle Stokes numbers, the particle velocity satisfies an extension of the Maxey–Riley equation. This equation contains several modifications to the original Maxey–Riley equation, some of which can be justifiably ignored under appropriate conditions (see, e.g., Michaelides[2] for a review).

The Maxey–Riley equation becomes singular in the limit of vanishing particle size. This singularity has prompted the use of geometric singular perturbation theory in the study of inertial particle motion in specific two-dimensional steady flows. Rubin, Jones, and Maxey[3] and Burns *et al.*[4] showed in specific flow models that inertial particle motion converges exponentially to a slow manifold in the phase space of the steady Maxey–Riley equation. Trajectories on this manifold move at a speed of $\mathcal{O}(\mathrm{St})$, with St denoting the particle Stokes number. The above authors also derived reduced equations of motion on the slow manifold in their specific examples. Mograbi and Bar-Ziv[5] outlined a general algorithm for determining such reduced equations for general steady flows and made qualitative observations about possible particle asymptotics.

Recently, Haller and Sapsis[6] showed that an exponentially attracting slow manifold exists for general unsteady inertial particle motion as long as the particle Stokes number is small enough. They also derived an explicit reduced equation on the slow manifold (*inertial equation*) that governs the asymptotic behavior of particles. In the case of neutrally buoyant particles (suspensions), the inertial equation coincides with the equations of Lagrangian particle motion. This would seem to imply that neutrally buoyant particles should synchronize exponentially fast with Lagrangian particle dynamics for small Stokes numbers.

By contrast, Babiano *et al.*[7] and Vilela *et al.*[8] give numerical evidence that two-dimensional suspensions do not approach Lagrangian particle motions; instead, their trajectories scatter around unstable manifolds of the Lagrangian particle dynamics. Szeri *et al.*[9] present specific examples of suspended microstructures in two-dimensional fluid flows where small changes of the modeling assumptions lead to drastically different dynamics. Babiano *et al.*[7] derive a criterion that characterizes the unstable regions in which scattering of inertial particles occurs. Their derivation follows an Okubo–Weiss-type heuristic reasoning, where it is assumed that the rate of change of the velocity gradient tensor calculated on a particle trajectory is small and hence can be neglected. However, as known counterexamples show (cf. Pierrehumbert and Yang[10] and Boffetta *et al.*[11]), such reasoning, in general, yields incorrect stability results except near fixed points of the flow field.

In the present work, we derive a rigorous analytical criterion for the stable and unstable regions of the slow manifold for weakly neutrally buoyant particles in general three-dimensional unsteady fluid flows. We also prove that the two-dimensional criterion of Babiano *et al.*[7] always gives a lower estimate for the exact domains of instability. Furthermore, we give an analytic bound for the Stokes number under which the slow manifold is always globally attracting. Finally, we illustrate our results on the Jung–Tél–Ziemniak[12] model of vortex shedding behind a cylinder in crossflow, as well as on the classic ABC flow.

II. SETUP

Let $\mathbf{u}(\mathbf{x},t)$ be the velocity field of a two- or three-dimensional fluid flow of density ρ_f. The fluid fills a compact (possibly time-varying) spatial region $\mathcal{D}$ with boundary $\partial\mathcal{D}$. We assume that $\mathbf{u}(\mathbf{x},t)$ and its first derivatives are continu-

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ously differentiable in their arguments. We denote the material derivative of the velocity field by

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\nabla \mathbf{u})\mathbf{u},$$

where ∇ denotes the gradient operator with respect to the spatial variable $\mathbf{x}$.

Let $\mathbf{x}(t)$ denote the path of a spherical particle p of density $\rho_p = \rho_f$ immersed in the fluid. Let the particle be spherical with radius $a \ll 1$, let L denote a characteristic length scale in the flow, let Re denote the Reynolds number, and let $\mathbf{v}(t) = \dot{\mathbf{x}}(t)$ be the Lagrangian velocity of the spherical particle. The particle satisfies the Maxey–Riley equation of motion (cf., e.g., Benczik, Toroczkai, and Tél[13] or Babiano *et al.*[7]),

$$\epsilon \dot{\mathbf{v}} - \epsilon \frac{D\mathbf{u}}{Dt} = -(\mathbf{v} - \mathbf{u}), \tag{1}$$

where

$$\epsilon = \frac{1}{\mu} \ll 1, \quad \mu = \frac{2}{3\text{St}}, \quad \text{St} = \frac{2}{9}\left(\frac{a}{L}\right)^2 \text{Re}.$$

Note that the larger the inertia parameter μ, the less significant the effect of inertia; in the $\mu \to \infty$ limit, Eq. (1) describes the motion of a passive ideal tracer particle.

Haller and Sapsis[6] proved that for $\epsilon > 0$ small enough, Eq. (1) admits a globally attracting invariant slow manifold. For neutrally buoyant particles, it has the form

$$M_\epsilon = \{(\mathbf{x}, \phi, \mathbf{v}) : \mathbf{v} = \mathbf{u}(\mathbf{x}, \phi)\}; \tag{2}$$

for non-neutrally-buoyant particles, M_ϵ is given by a graph, $\mathbf{v} = \mathbf{u}(\mathbf{x}, \phi) + \mathcal{O}(\epsilon)$.

The dynamics on M_ϵ is governed by the reduced Maxey–Riley equations (*inertial equation*),

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t), \tag{3}$$

i.e., by the equation of motion for infinitesimal fluid elements.

Using an observation of Babiano *et al.*,[7] we can conclude that the invariant manifold M_ϵ and the corresponding reduced equation (3) exist for all values of ϵ in the neutrally buoyant case. Specifically, Eq. (1) is equivalent to

$$\dot{\mathbf{v}} - \mathbf{u}_t - (\nabla \mathbf{u})\mathbf{u} = -\mu(\mathbf{v} - \mathbf{u}).$$

As pointed out by Babiano *et al.*,[7] this last equation can be recast in the form

$$\dot{\mathbf{v}} - \mathbf{u}_t - (\nabla \mathbf{u})\mathbf{v} = -\mu(\mathbf{v} - \mathbf{u}) + (\nabla \mathbf{u})(\mathbf{u} - \mathbf{v}),$$

or, equivalently,

$$\begin{aligned} &\frac{d}{dt}[\mathbf{v} - \mathbf{u}(\mathbf{x}, t)] = -[\nabla \mathbf{u}(\mathbf{x}, t) + \mu \mathbf{I}][\mathbf{v} - \mathbf{u}(\mathbf{x}, t)], \\ &\frac{d}{dt}\mathbf{x} = \mathbf{v}. \end{aligned} \tag{4}$$

This shows that M_ϵ defined in Eq. (2) is an invariant manifold for any $\mu = 1/\epsilon$.

III. GLOBAL ATTRACTIVITY OF THE SLOW MANIFOLD

While M_ϵ exists for any $\epsilon > 0$, it is not guaranteed to be globally attracting for larger values of ϵ (i.e., for smaller values of μ). Here we give a sufficient condition under which the global attractivity of M_ϵ is guaranteed.

Theorem 1. *Assume that for some fixed* $\epsilon > 0$, *the smallest eigenvalue field* $\lambda_{\min}(\mathbf{x}, t)$ *of the symmetric tensor field* $\mathbf{I} + \epsilon \mathbf{S}(\mathbf{x}, t)$ *is uniformly positive for all* $\mathbf{x} \in \mathcal{D}$ *and* $t \in \mathbb{R}^+$. *Then the invariant manifold* M_ϵ *is globally attracting, i.e., all neutrally buoyant particle motions synchronize exponentially fast with infinitesimal Lagrangian fluid trajectories.*

Proof: Applying the change of coordinates $\mathbf{z} = \mathbf{v} - \mathbf{u}(\mathbf{x}, t)$ to Eq. (4), we obtain the system

$$\begin{aligned} &\dot{\mathbf{z}} = -[\nabla \mathbf{u}(\mathbf{x}, t) + \mu \mathbf{I}]\mathbf{z}, \\ &\dot{\mathbf{x}} = \mathbf{z} + \mathbf{u}(\mathbf{x}, t). \end{aligned} \tag{5}$$

Note that the $\{\mathbf{z} = 0\}$ invariant subspace of this equation corresponds to the invariant manifold M_ϵ for any $\mu = 1/\epsilon$. We fix a solution $[\mathbf{x}(t), \mathbf{z}(t)]$ of Eq. (5), substitute the solution into Eq. (5), and multiply the $\mathbf{z}$ component of the resulting equation by $\mathbf{z}(t)$ to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\mathbf{z}|^2 &= -\langle \mathbf{z}, \{\nabla \mathbf{u}[\mathbf{x}(t), t] + \mu \mathbf{I}\}\mathbf{z}\rangle \\ &= \langle \mathbf{z}, [-\mathbf{S}(\mathbf{x}(t), t) - \mu \mathbf{I}]\mathbf{z}\rangle \\ &\leq \lambda_{\max}[-\mathbf{S}(\mathbf{x}(t), t) - \mu \mathbf{I}]|\mathbf{z}|^2. \end{aligned} \tag{6}$$

Here we have introduced the rate-of-strain tensor

$$\mathbf{S} = \tfrac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] \tag{7}$$

and used the notation $\lambda_{\max}[T]$ to refer to the maximal eigenvalue of a tensor T. Integration of Eq. (6) yields

$$|\mathbf{z}(t)| \leq |\mathbf{z}(t_0)| e^{-\int_{t_0}^{t} \lambda_{\min}[\mathbf{S}(\mathbf{x}(s), s) + \mu \mathbf{I}] ds},$$

with $\lambda_{\min}$ referring to the minimal eigenvalue of the appropriate tensor. We conclude that the $\mathbf{z} = 0$ subspace is globally attracting if

$$\lim_{t \to \infty} \int_{t_0}^{t} \lambda_{\min}[\mathbf{I} + \epsilon \mathbf{S}(\mathbf{x}(s; \mathbf{x}_0), s)] ds = \infty$$

for all $\mathbf{x}_0 \in \mathcal{D}$. This is certainly satisfied if

$$\lambda_{\min}[\mathbf{I} + \epsilon \mathbf{S}(\mathbf{x}, t)] \geq c > 0$$

for some positive constant c and for all $\mathbf{x} \in \mathcal{D}$ and $t \in \mathbb{R}$. □

For two-dimensional incompressible flows, $\lambda_{\min}[\mathbf{I} + \epsilon \mathbf{S}]$ is the smaller root of the characteristic equation

$$\lambda^2 - 2\lambda + (1 + \epsilon^2 \det \mathbf{S}) = 0.$$

Therefore, the two-dimensional version of the sufficient condition in Theorem 1 requires that

$$\lambda_{\min} = 1 - \epsilon\sqrt{-\det \mathbf{S}(\mathbf{x}, t)} > 0,$$

or, equivalently,

$$\mu - \sqrt{|\det \mathbf{S}(\mathbf{x},t)|} > 0 \tag{8}$$

holds uniformly for all $\mathbf{x} \in \mathcal{D}$ and $t \in \mathbb{R}$.

IV. LOCAL DIVERGENCE ALONG THE SLOW MANIFOLD

Even if the above sufficient condition fails to hold, M_ϵ can still be globally attractive, although trajectories may temporarily diverge from M_ϵ while they pass by regions of divergence on M_ϵ. As the following result shows, these regions of local divergence are precisely the domains where the sufficient condition of Theorem 1 is violated.

Theorem 2. *For any* $\epsilon > 0$, *the invariant manifold* M_ϵ *repels all close enough trajectories* $[\mathbf{x}(t), \mathbf{v}(t)]$ *as long as they satisfy*

$$\lambda_{\min}[\mathbf{S}(\mathbf{x}(t),t) + \mu\mathbf{I}] < 0. \tag{9}$$

Proof: For a fixed initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, let $\mathbf{\Phi}_{t_0}^t$ denote the fundamental matrix solution of the first equation of Eq. (5). By definition, we then have

$$\dot{\mathbf{\Phi}}_{t_0}^t = \mathbf{A}(t)\mathbf{\Phi}_{t_0}^t,$$
$$\mathbf{A}(t) = -[\nabla \mathbf{u}(\mathbf{x}(t;t_0,\mathbf{x}_0),t) + \mu\mathbf{I}]. \tag{10}$$

We suppress the dependence of $\mathbf{A}$ on $\mathbf{x}_0$ and t_0 for notational simplicity.

Observing that $\mathbf{z}(t;t_0,\mathbf{x}_0,\mathbf{z}_0) = \mathbf{\Phi}_{t_0}^t \mathbf{z}_0$, we find that

$$|\mathbf{z}(t;t_0,\mathbf{x}_0,\mathbf{z}_0)|^2 = \langle \mathbf{\Phi}_{t_0}^t \mathbf{z}_0, \mathbf{\Phi}_{t_0}^t \mathbf{z}_0 \rangle = \langle \mathbf{z}_0, (\mathbf{\Phi}_{t_0}^t)^T \mathbf{\Phi}_{t_0}^t \mathbf{z}_0 \rangle.$$

Therefore, $\mathbf{z}_0$ perturbations to the manifold M_ϵ will grow or decay depending on the eigenvalue configuration of the Cauchy–Green strain tensor $(\mathbf{\Phi}_{t_0}^t)^T \mathbf{\Phi}_{t_0}^t$. Specifically, if $(\mathbf{\Phi}_{t_0}^t)^T \mathbf{\Phi}_{t_0}^t$ has an eigenvalue larger than 1 for a given t, typical $\mathbf{z}_0$ perturbations to M_ϵ will increase in norm over the interval $[t_0, t]$.

Our interest is finding locations of instantaneous growth in the $\mathbf{z}$ direction at points of the manifold M_ϵ. By instantaneous growth, we mean growth in the limit of $t \to t_0$. Note that in that limit, all eigenvalues of $(\mathbf{\Phi}_{t_0}^t)^T \mathbf{\Phi}_{t_0}^t$ tend to 1, thus instantaneous growth is governed by how the unit eigenvalues of $(\mathbf{\Phi}_{t_0}^t)^T \mathbf{\Phi}_{t_0}^t$ perturb away from unity as t is increased from t_0. To see this, we will calculate the instantaneous stability indicator

$$\sigma(\mathbf{x}_0, t_0) = \lim_{t \to t_0} \frac{1}{t - t_0} \log \lambda_{\max}[(\mathbf{\Phi}_{t_0}^t)^T \mathbf{\Phi}_{t_0}^t], \tag{11}$$

which is a Lyapunov-exponent-type quantity, except that the limit is taken at $t \to t_0$, as opposed to $t \to \infty$.

Since we have assumed that $\mathbf{u}(\mathbf{x}, t)$ is twice continuously differentiable, standard regularity results for ordinary differential equations guarantee that $\mathbf{\Phi}_{t_0}^t$ is also twice continuously differentiable. Thus, we can expand $\mathbf{\Phi}_{t_0}^t$ with respect to time into a Taylor series of the form

$$\mathbf{\Phi}_{t_0}^t = \mathbf{I} + \dot{\mathbf{\Phi}}_{t_0}^t\big|_{t=t_0} \Delta t + \mathcal{O}[(\Delta t)^2],$$

where $\Delta t = t - t_0$. Using Eq. (10), we obtain

$$\mathbf{\Phi}_{t_0}^t = \mathbf{I} + \mathbf{A}(t)\Delta t + \mathcal{O}[(\Delta t)^2].$$

For the Cauchy–Green strain tensor

$$\mathbf{B}(t;t_0) = (\mathbf{\Phi}_{t_0}^t)^T \mathbf{\Phi}_{t_0}^t,$$

we obtain the expression

$$\mathbf{B}(t;t_0) = \mathbf{I} + [\mathbf{A}^T(t) + \mathbf{A}(t)]\Delta t + \mathcal{O}[(\Delta t)^2].$$

Let $\phi_i(t)$ and $\mathbf{e}_i(t)$, $i = 1, \ldots n$ denote the real eigenvalues and the corresponding eigenvectors, respectively, of the symmetric tensor $\mathbf{B}(t;t_0)$. Because $\mathbf{B}(t_0;t_0) = \mathbf{I}$ is semisimple, its eigenvalues $\phi_i(t)$ and eigenvectors $\mathbf{e}_i(t)$ will be differentiable with respect to time at $t = t_0$ (see, e.g., Lancaster,[14] Theorem 7). Thus, by expanding the eigenvalue problem

$$\mathbf{B}(t;t_0)\mathbf{e}_i(t) = \phi_i(t)\mathbf{e}_i(t)$$

in a Taylor series in Δt, we obtain

$$[\mathbf{I} + [\mathbf{A}^T(t_0) + \mathbf{A}(t_0)]\Delta t][\mathbf{e}_i(t_0) + \dot{\mathbf{e}}_i(t_0)\Delta t]$$
$$= [1 + \dot{\phi}_i(t_0)\Delta t][\mathbf{e}_i(t_0) + \dot{\mathbf{e}}_i(t_0)\Delta t] + \mathcal{O}(\Delta t^2).$$

Equating $\mathcal{O}(\Delta t)$ terms in this last equation, we obtain

$$[\mathbf{A}^T(t_0) + \mathbf{A}(t_0)]\mathbf{e}_i(t_0) = \dot{\phi}_i(t_0)\mathbf{e}_i(t_0).$$

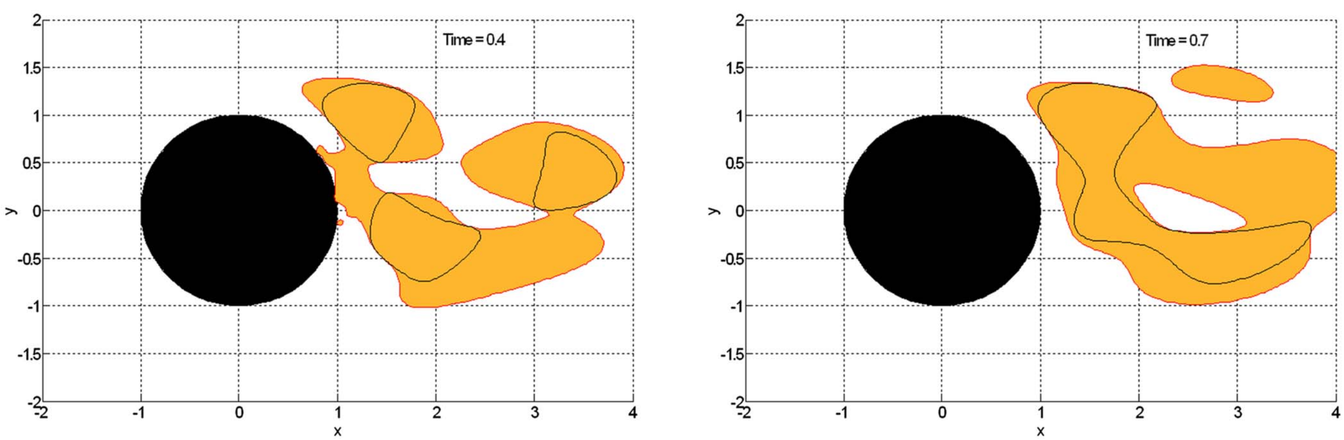

FIG. 1. (Color online) Exact domain of neutrally buoyant particles' divergence [shaded (yellow online) regions] and its lower estimate (encircled by black curves) by the criterion of Babiano *et al.* for two different times.

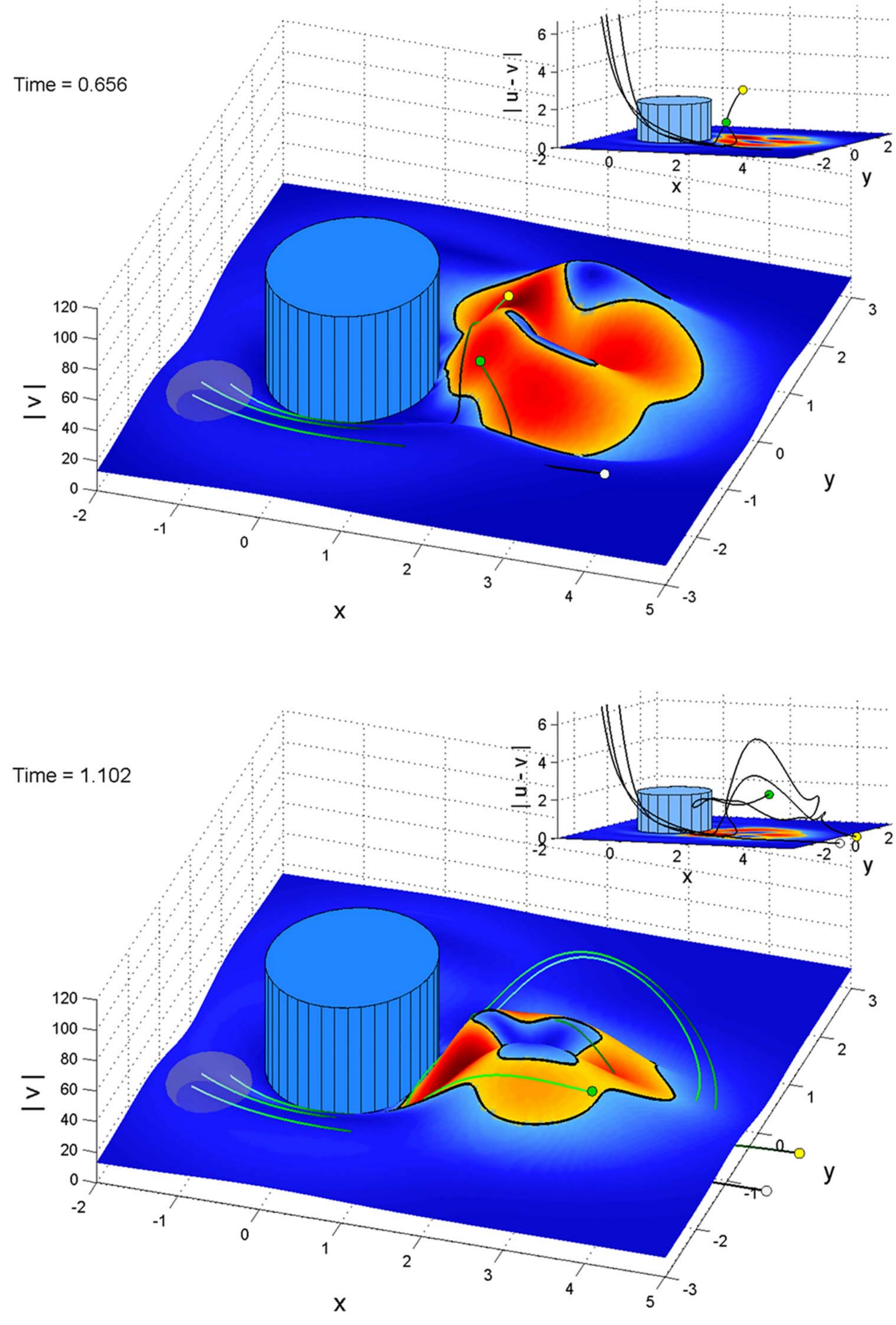

FIG. 2. (Color online) Global convergence of neutrally buoyant particles to the slow manifold M_ϵ, interrupted by regions of divergence along the slow manifold. The repelling domain on the slow manifold, encircled by the black solid curves (red regions online), satisfies formula (13).

To compute $\sigma(\mathbf{x}_0,t_0)$ from Eq. (11), we note that

$$\lambda_{\max}[(\mathbf{\Phi}_{t_0}^t)^T\mathbf{\Phi}_{t_0}^t] = \lambda_{\max}\{\mathbf{I} + [\mathbf{A}^T(t) + \mathbf{A}(t)]\Delta t + \mathcal{O}[(\Delta t)^2]\}$$
$$= 1 + \lambda_{\max}[\mathbf{A}^T(t) + \mathbf{A}(t)]\Delta t + \mathcal{O}[(\Delta t)^2].$$

Then, expanding the logarithm of $\lambda_{\max}[(\mathbf{\Phi}_{t_0}^t)^T\mathbf{\Phi}_{t_0}^t]$ into a Taylor series, we obtain

$$\log \lambda_{\max}[(\mathbf{\Phi}_{t_0}^t)^T\mathbf{\Phi}_{t_0}^t] = \lambda_{\max}[\mathbf{A}^T(t) + \mathbf{A}(t)]\Delta t + \mathcal{O}[(\Delta t)^2].$$

Thus, Eqs. (7), (10), and (11) give

$$\sigma(\mathbf{x}_0,t_0) = \lambda_{\max}[\mathbf{A}^T(t) + \mathbf{A}(t)]$$
$$= 2\lambda_{\max}[-\mathbf{S}(\mathbf{x}_0,t_0) - \mu\mathbf{I}]$$
$$= -2\lambda_{\min}[\mathbf{S}(\mathbf{x}_0,t_0) + \mu\mathbf{I}].$$

Regions of divergence on M_ϵ satisfy $\sigma(\mathbf{x}_0,t_0) > 0$, or equivalently,

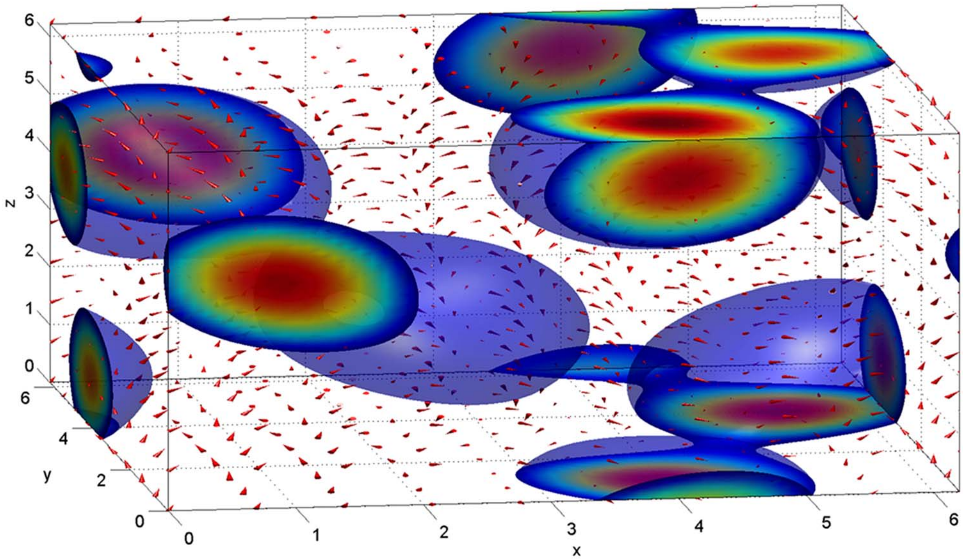

FIG. 3. (Color online) Regions of divergence in the ABC flow with parameters $A=12$, $B=15$, $C=10$, and inverse Stokes number $\epsilon=0.075$.

$$\lambda_{\min}[\mathbf{S}(\mathbf{x}_0,t_0)+\mu\mathbf{I}]<0.$$

Replacing the initial location and time $(\mathbf{x}_0,t_0)$ with $(\mathbf{x}(t),t)$, we obtain the statement of Theorem 2. □

The proof of Theorem 2 shows that domains of M_ϵ with indefinite $\mathbf{S}(\mathbf{x},t)+\mu\mathbf{I}$ repel nearby trajectories off M_ϵ. By contrast, domains of M_ϵ with positive definite $\mathbf{S}(\mathbf{x},t)+\mu\mathbf{I}$ attract nearby trajectories. Lines along which $\mathbf{S}(\mathbf{x},t)+\mu\mathbf{I}$ is semidefinite separate regions of attraction and repulsion from each other on M_ϵ. For incompressible flows, $\mathbf{S}(\mathbf{x},t)$ must have zero trace, and hence $\mathbf{S}(\mathbf{x},t)+\mu\mathbf{I}$ cannot be negative definite.

At first sight, Theorems 1 and 2 might appear to imply each other but they are, in fact, independent results. The first theorem gives a sufficient condition for global attractivity of the slow manifold, whereas the second theorem gives a sufficient condition for local divergence along the same manifold. Note that a lack of local linear instabilities does not imply global attraction for the manifold. Conversely, just because a specific sufficient condition for global attraction is violated, we cannot conclude that there are necessarily regions of local divergence along the manifold.

For two-dimensional fluid flows, condition (9) can be written explicitly as

$$\mathrm{Tr}[\mathbf{S}(\mathbf{x}_0,t_0)+\mu\mathbf{I}] - \sqrt{\mathrm{Tr}[\mathbf{S}(\mathbf{x}_0,t_0)+\mu\mathbf{I}]^2-4\det[\mathbf{S}(\mathbf{x}_0,t_0)+\mu\mathbf{I}]}<0,$$

or, equivalently,

$$-\mathrm{Tr}[\mathbf{S}(\mathbf{x}_0,t_0)]+\sqrt{s_1^2+s_2^2}>2\mu, \tag{12}$$

where $s_1=\partial_x u_x-\partial_y u_y$ and $s_2=\partial_y u_x+\partial_x u_y$ denoting the normal and shear strain components, respectively. For incompressible flows, Eq. (12) further simplifies to

$$-\det \mathbf{S}(\mathbf{x},t)>\mu^2. \tag{13}$$

By contrast, for two-dimensional incompressible flows, Babiano *et al.*[7] use an Okubo–Weiss-type reasoning to obtain the instability condition

$$-\det \nabla\mathbf{u}(\mathbf{x},t)>\mu^2. \tag{14}$$

We note that

$$\det \nabla\mathbf{u}(\mathbf{x},t)=\det\mathbf{S}(\mathbf{x},t)+\frac{\omega^2}{4}\geqslant\det\mathbf{S}(\mathbf{x},t),$$

where $\omega^2=(\partial_y u_x-\partial_x u_y)^2$ is the squared vorticity. As a result, we have

$$-\det\mathbf{S}(\mathbf{x},t)\geqslant-\det\nabla\mathbf{u}(\mathbf{x},t)>\mu^2, \tag{15}$$

therefore the criterion (14) will typically underestimate the domains satisfying the exact divergence criterion (13).

V. TWO EXAMPLES

A. von Kármán vortex street

To illustrate our results, we consider a two-dimensional model of the von Kármán vortex street in the wake of a cylinder by Jung, Tél, and Ziemniak.[12] Since the flow is incompressible, a stream function $\Psi=[0,0,\Psi(x,y,t)]$ can be defined such that the fluid velocity will be given by $\mathbf{u}=\nabla\times\Psi$. Jung, Tél, and Ziemniak[12] give an explicit expression for Ψ and determine a set of parameter values for which the model shows good agreement with Navier–Stokes simulations of vortex shedding behind a cylinder at Reynolds numbers $\mathrm{Re}\simeq 250$. One of the parameters entering the model's stream function is the average strength of the detaching vortices, w, which we take to be $w=8\times 24/\pi$ following Benczik, Toroczkai, and Tél.[13] For our inertia parameter, we select $\epsilon=1/\mu=10^{-2}$. In our computation of inertial particle trajectories, we use a fourth-order Runge–Kutta algorithm with absolute integration tolerance 10^{-7}.

First, in Fig. 1, we present a comparison of the exact domain of divergence on the slow manifold (shaded region) with the domain predicted by the criterion (14) of Babiano *et al.*[7] (region encircled by black curve) for two different times. As predicted by formula (15), the latter region is always

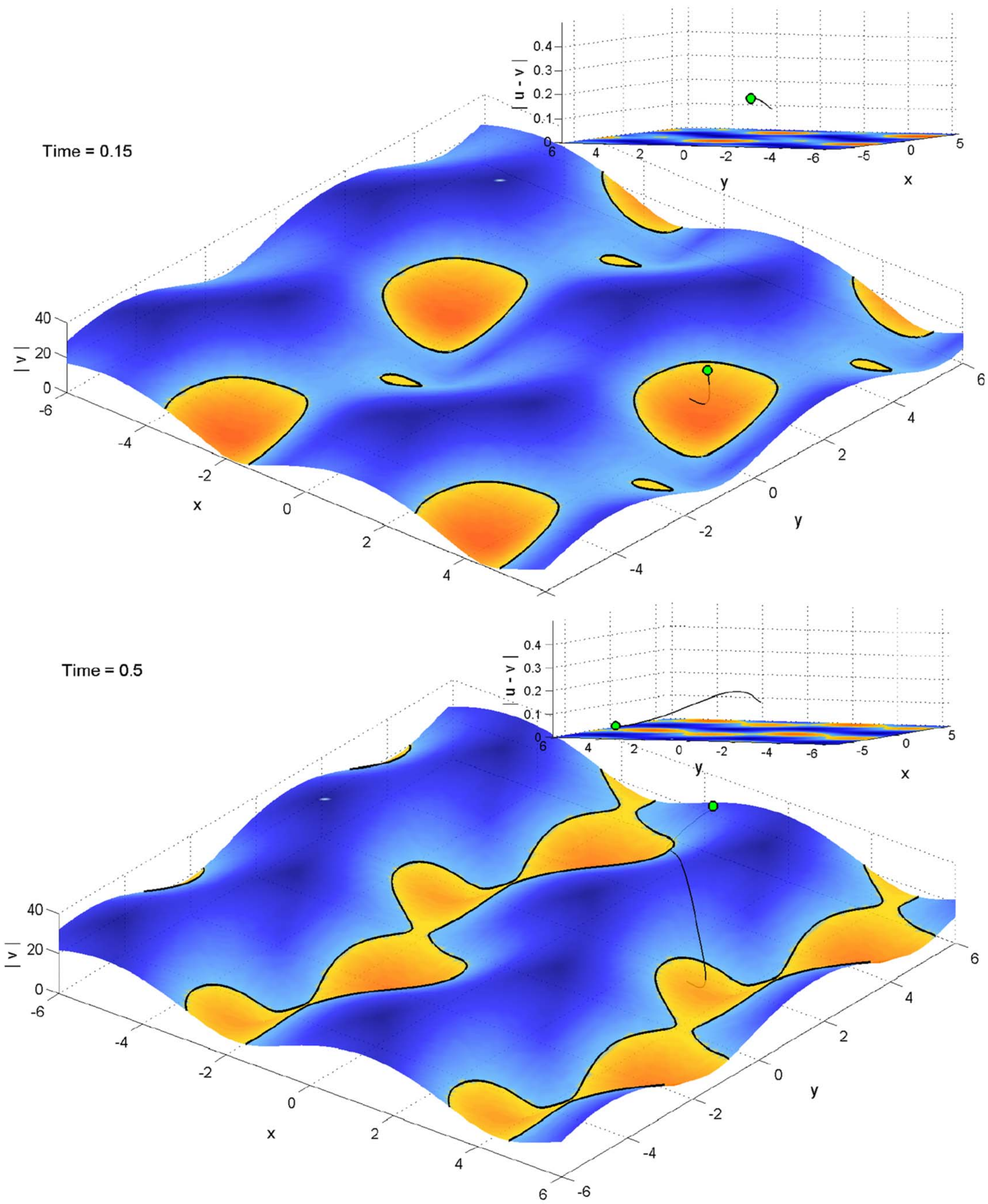

FIG. 4. (Color online) Global convergence of a neutrally buoyant particle to the slow manifold M_ϵ [computed for each time at the instantaneous vertical particle position $z_p(t)$] interrupted by regions of divergence along the slow manifold. The repelling domain on the slow manifold, encircled by the black solid curves (red regions online), satisfies formula (9).

inside the former region, and hence the criterion of Babiano *et al.* underestimates the region of divergence, sometimes even missing full connected components.

In Fig. 2, we present two snapshots of inertial particle dynamics viewed in the space of $(x, y, |\mathbf{v}|)$. We show the instantaneous slow manifold M_ϵ as a surface, at two different times; regions encircled by the black solid curves (red regions online) show the domain of local divergence on the slow manifold. We release neutrally buoyant particles from a gray area of the phase space and track their evolution. The smaller subplots in the figure show the distance $|\mathbf{u}-\mathbf{v}|$ of the particles from the slow manifold (in these plots, therefore, the slow manifold appears as a plane). Note that while the particles end up converging to the slow manifold, they are temporarily repelled by the domain of local divergence (regions encircled by the black solid curves) as predicted by our theory.

B. ABC flow

For the illustration of our results in three dimensions, we consider the Arnold–Beltrami–Childress (ABC) flow described by the steady velocity field

$$\mathbf{u}(x,y,z) = \begin{pmatrix} A \sin z + C \cos y \\ B \sin x + A \cos z \\ C \sin y + B \cos x \end{pmatrix}. \quad (16)$$

This three-parameter family of spatially periodic flows provides a simple steady-state solution of Euler's equations, as shown by Arnold.[15] For the flow parameters, we choose $A=12$, $B=15$, $C=10$; for our inertia parameter, we select $\epsilon=1/\mu=0.075$. In our computation of inertial particle trajectories, we again use a fourth-order Runge–Kutta algorithm with absolute integration tolerance 10^{-7}.

In Fig. 3, we show the unstable regions of the flow satisfying Eq. (9). The coloring inside these domains illustrates how the stability indicator $\lambda_{\min}[\mathbf{S}(\mathbf{x}_0,t_0)+\mu\mathbf{I}]$ varies inside the unstable volumes. In the same figure, the arrows show the velocity vectors (16).

In Fig. 4, we present two snapshots of inertial particle dynamics viewed in the space $\{x,y,|\mathbf{v}[\cdot,\cdot,z_p(t)]|\}$, where $z_p(t)$ is the instantaneous z coordinate of the particle. We show the slow manifold M_ϵ computed for each time at the instantaneous vertical particle position $z_p(t)$; regions encircled by the black solid curves (red regions online) show the domain of local divergence on the slow manifold. The smaller subplots in the figure show the distance $|\mathbf{u}-\mathbf{v}|$ of the particles from the slow manifold. Note how particles are repelled by the slow manifold over the domain of local divergence, while particles converge to the slow manifold over the regions that are outside of the black curves (blue regions online).

VI. CONCLUSIONS

We have derived an analytic criterion for regions of divergence on the slow manifold that governs the asymptotic motion of neutrally buoyant particles in a general unsteady fluid flow. We have also shown that an earlier formula for these unstable regions by Babiano *et al.*[7] always gives a lower estimate. Furthermore, we have derived a sufficient criterion for the global attractivity of the slow manifold. Under this condition, all inertial particle motions synchronize with Lagrangian fluid motion. We illustrated our results on inertial particle dynamics in a two-dimensional model of the von Kármán vortex street in the wake of a cylinder by Jung, Tél, and Ziemniak[12] and in the classic three-dimensional steady ABC flow.

Studying the topology of regions of divergence on the slow manifold for three-dimensional flows requires further work. An application of the present results to identifying regions of inaccuracies for particle image velocimetry will be given elsewhere.

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