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Diffusion at intersecting resonances in Hamiltonian systems

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Abstract

We study n-degree-of-freedom, nearly integrable Hamiltonian systems near the intersection of a stronger and a weaker resonance. We construct motions that cross the weaker resonance along the stronger one at a speed larger than that of Arnold diffusion. We also derive estimates for the measure of such solutions.

1. Introduction

In this note we consider nearly integrable, n-degree-of-freedom Hamiltonian systems of the form

$$H(I,\phi;\epsilon) = H_0(I) + \epsilon H_1(I,\phi;\epsilon), \qquad (1)$$

where $(I,\phi) \in \mathbb{R}^n \times \mathbb{T}^n$ are action-angle variables with $n \geqslant 3$, $0 < \epsilon \ll 1$ is a small parameter, and H_0 and H_1 are analytic functions. It is well known that the unperturbed flow generated by the Hamiltonian H_0 is completely integrable with the n independent integrals $I_1, \ldots, I_n$. The phase space of this integrable system is foliated by n-dimensional invariant tori of the form I = const. The KAM theorem [1] states that in the generic case most of these tori survive in the perturbed system. The surviving tori are precisely those that are sufficiently well separated from the *resonance web*,

$$W = \{I \subset U | \langle D_I H_0(I), k\rangle = 0,\ k \in \mathbb{Z}^n\}. \qquad (2)$$

For two-degree-of-freedom systems that satisfy the condition of isoenergetic nondegeneracy [1], the KAM theory actually guarantees that all motions on each three-dimensional energy surface H_0 = const are either quasiperiodic or remain trapped forever between two adjacent surviving two-tori. However, if $n \geqslant 3$, the surviving tori do not serve as barriers to nonquasiperiodic motions and large changes in the initial action values become possible for trajectories that traverse in a neighborhood of the resonance web W.

In his famous paper [2], Arnold sketched an example of a nearly integrable system which exhibits an O(1) variation in the action values even in the limit $\epsilon \to 0$. He proposed that the fundamental mechanism for such an evolution is provided by the transverse intersection of stable and unstable manifolds (or "whiskers") of a chain of lower dimensional invariant tori that is created in the destruction of unperturbed n-tori along a given resonance. This phenomenon has become known as *Arnold diffusion,* and it is believed to be the underlying cause for long-time instability in multi-degree-of-freedom Hamiltonian systems [3,4]. By Nekhorosev's theorem [1], the *average* speed of this diffusion is necessarily exponentially small in the generic case.

Although in the physics literature the existence of Arnold diffusion is sometimes treated as a fact,

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the mathematical details of Arnold's example are still not fully understood [5]. Recently, in an important paper [6], Chierchia and Gallavotti clarified the details of the construction of transition chains of tori for a special class of problems that involves the D'Alembert precession-nutation model. At the same time, their work further emphasized the fact (see also Ref. [5]) that the general picture of diffusion proposed by Arnold can be justified only away from multiple resonances or *resonance junctions.* Since in systems with at least three-degrees-of-freedom multiple resonances form a dense subset of the action space, their role cannot be ignored [7]. There is also increasing numerical evidence [8] (see also Refs. [3,4]) for symplectic maps, that the speed of diffusion of the action variables becomes much larger near resonance junctions than that of Arnold diffusion along single resonances. In particular, a characteristic *cross-resonance diffusion* appears to happen on time scales much shorter than exponential and it curiously involves higher order resonances which are usually believed to be less significant [8].

In this paper, motivated by the above numerical observations by Laskar, we study the geometry and dynamics near a multiplicity two resonance in the phase space under the assumption that *one of the two resonances is weaker than the other one.* However, our results can easily be extended to the intersection of two nearly equally strong resonances provided the perturbation contains one of the resonant harmonics with much smaller amplitude than the other. This latter case is a generalization of the example studied by Benettin and Gallavotti [9].

The key observation in our study is that weak–strong resonance junctions admit a near-integrable dynamics which enables one to study details of their geometry. In particular, we describe how motions pass through the weaker resonance guided by the stronger resonance. This passage happens on a weakly hyperbolic invariant manifold of $(n-1)$-dimensional whiskered tori. We establish an order $O(1/\sqrt{\epsilon})$ upper estimate for the time of the passage which shows that cross-resonance diffusion is indeed much faster than the exponentially slow Arnold diffusion. Using a result of Fenichel [11] from geometric singular perturbation theory, we extend the same result for an open set around the original set of $(n-1)$-tori. It turns out that the measure of this set can be taken *algebraic* in the perturbation parameter ϵ. We conclude the paper by comparing diffusion near weak–strong resonance junctions to Arnold diffusion and describing some further results on a special type of chaotic dynamics near multiple resonances. For full proofs and more details of the present results, the reader is referred to Ref. [11] where the intersection of more than two resonances is also treated.

2. Normal form for weak–strong resonance junctions

Throughout this paper we shall assume that for some fixed constant $\sigma > 0$, the complex extension $H_1(I, z; \epsilon)$ of the Hamiltonian H_1 is analytic in the domain $|\mathrm{Im}\ z_i| \leqslant \sigma$ of the $2n$-dimensional complex space $\mathbb{C}^n$, and for some bounded set $S \subset \mathbb{R}^n$,

$$\sup_{I\in S,\ |\mathrm{Im}\ z_i|\leqslant\sigma} |H_1| < K_\sigma$$

holds with an appropriate positive constant K_σ and for ϵ sufficiently small. We shall also need the Fourier expansion of H_1, which can be written in the form

$$H_1(I, \phi; \epsilon) = \sum_{k\in\mathbb{Z}^n} h_k(I; \epsilon) \exp(\mathrm{i}\langle k, \phi\rangle).$$

Our basic assumption is that the frequencies of the unperturbed Hamiltonian H_0 satisfy precisely two independent resonance relationships at some point I^r of the action space, i.e., there exist two linearly independent integer vectors $l, m \in \mathbb{Z}^n$ such that

$$\langle l, D_I H_0(I^r)\rangle = \langle m, D_I H_0(I^r)\rangle = 0.$$

We assume that the first resonance is "forced" by the perturbation at leading order, i.e., $\inf_{I\in S} |h_l(I; 0)| > c_0 > 0$. We denote the resonant module generated by the vectors l and m by M, i.e., we let

$$M = \{k \in \mathbb{Z}^n \mid k = c_1 l + c_2 m,\ c_1, c_2 \in \mathbb{Z}\}.$$

Let $r_1 \in M$ be a "minimal" element of the module M, i.e., an integer vector with the property that for any $k \in M$, $|k| \geqslant |r_1|$ holds. Also, let r_2 be an element of M which is linearly independent of r_1 and its modulus is minimal among all elements of M that are linearly independent of r_1. Then, it is always possible to find linearly independent integer vectors $r_3, \dots, r_n$ such that for the $n \times n$ matrix

$$T = \begin{pmatrix} r_1^t \\ r_2^t \\ \vdots \\ r_n^t \end{pmatrix}$$

we have $\det T = 1$ [1,9]. We then introduce the change of variables [1]

$$\psi = T\phi, \quad \sqrt{\epsilon}\, J = (T^t)^{-1}(I - I^r), \tag{3}$$

which is only canonical up to the factor $\sqrt{\epsilon}$, thus we have to divide the transformed Hamiltonian $H = H_0 + \epsilon H_1$ by this factor to preserve the corresponding Hamiltonian equations. Then Taylor expanding the rescaled H_0 and H_1 at the resonant action value I^r and dropping the constant $H_0(I^r)/\sqrt{\epsilon}$ gives the expressions

$$\begin{aligned} H_0(J;\epsilon) &= \langle TD_I H_0(I^r), J\rangle \\ &\quad + \tfrac{1}{2}\sqrt{\epsilon}\, \langle J, TD_I^2 H_0(I^r) T^t J\rangle + \mathrm{O}(\epsilon), \\ \sqrt{\epsilon}\, H_1(J,\psi;\epsilon) &= \sqrt{\epsilon}\Bigg(\sum_{p\in\mathbb{Z}^2} \tilde{h}_p \exp[\mathrm{i}\langle p, (\psi_1,\psi_2)\rangle] \\ &\quad + \sum_{k\in\mathbb{Z}^n - M} h_k(I^r;0)\exp(\mathrm{i}\langle k,\psi\rangle) + \mathrm{O}(\sqrt{\epsilon})\Bigg), \end{aligned}$$

where $\tilde{h}_p = h_k(I^r;0)$ whenever $k = p_1 r_1 + p_2 r_2$ (here p_j denotes the jth element of the integer vector $p \in \mathbb{Z}^2$). Note that we separated the resonant and nonresonant combinations of the phases in the expression of H. As a result, the first sum in the above expression only depends on the *slow angles* ψ_1 and ψ_2, whereas all harmonics in the second sum do depend explicitly on integer multiples of the *fast angles* $\psi_3, \dots, \psi_n$ and hence have zero averages with respect to these variables. Since, by our original assumption, the frequencies corresponding to these fast angles do not satisfy any resonance relationship in a neighborhood of the point $J = 0$, the method of multi-phase averaging [12] guarantees the existence of a near-identity, canonical change of variables that transforms the explicit fast angle dependence of H to terms of order $\mathrm{O}(\epsilon)$. In fact, in a small but fixed neighborhood of $J = 0$ further higher order averaging transformations can be constructed which push the fast angle dependence of H_1 to terms that are exponentially small in ϵ [9,13]. Thus, splitting our slow and fast variables into the new variables

$$A = (J_1, J_2), \quad \alpha = (\psi_1, \psi_2),$$
$$B = (\sqrt{\epsilon}\, J_3, \dots, \sqrt{\epsilon}\, J_n), \quad \beta = (\psi_3, \dots, \psi_n),$$

and retaining the same notation for B and β after performing the averaging transformations, we obtain the normal form Hamiltonian

$$\begin{aligned} H(A,\alpha,B,\beta;\epsilon) &= \sqrt{\epsilon}\, \langle b, B\rangle \\ &\quad + \sqrt{\epsilon}\, [H_{\mathrm{pend}}(A,\alpha) + \sqrt{\epsilon}\, H_2(A,\alpha,B;\sqrt{\epsilon})] \\ &\quad + \mathrm{e}^{-c/\epsilon} H_3(A,\alpha,B,\beta;\sqrt{\epsilon}), \end{aligned} \tag{4}$$

with

$$H_{\mathrm{pend}}(A,\alpha) = \tfrac{1}{2}\langle A, PA\rangle + \sum_{p\in\mathbb{Z}^2} \tilde{h}_p \exp(\mathrm{i}\langle p,\alpha\rangle). \tag{5}$$

Here $b \in \mathbb{R}^{n-2}$ contains the last $n-2$ elements of the vector $TD_I H_0(I^r)$, the symmetric matrix $P \in \mathbb{R}^{2\times 2}$ is the first 2×2 minor of the matrix $TD_I^2 H_0(I^r)T^t$, the functions H_2 and H_3 are analytic in their arguments, $c > 0$ is an appropriate constant and H_{pend} is the well-know pendulum-type Hamiltonian. The new symplectic form corresponding to this Hamiltonian is $\omega = \mathrm{d}\alpha \wedge \mathrm{d}A + \sqrt{\epsilon}\,\mathrm{d}\beta \wedge \mathrm{d}B$, hence the (β, B) equations in the associated Hamiltonian vector field are multiplied by a factor of $1/\sqrt{\epsilon}$ compared to the usual canonical equations.

As one can directly read off from (4), the action-type quantities $B_1, \dots, B_{n-2}$ are conserved quantities for the normal form if we neglect the exponentially small terms. This fact enables us to treat the corresponding normalized Hamiltonian vector field as an exponentially small perturbation of the terms that do not depend on the fast phases β. Since in the β-independent Hamiltonian system the B-components of the equations decouple, we naturally arrive at the study of a two-degree-of-freedom Hamiltonian which is an order $\mathrm{O}(\sqrt{\epsilon})$ perturbation of $H_{\mathrm{pend}}(A,\alpha)$.

We are now in the position to formulate our major hypothesis that I^r lies in the intersection of a weaker and a stronger resonance, corresponding to the integer vectors r_2 and r_1, respectively. We require the

p_2-dependent part of the Fourier series in (5) to be smaller than a small parameter μ with $0 < \mu \ll \|H_1(I^r, \phi; \epsilon)\|$. This assumption is reasonable since H_1 is as analytic function so its Fourier coefficients decay exponentially. In particular, for any $0 < \kappa < \sigma$ we have the estimate [12]

$$\left\| \sum_{|k| \geqslant |r_2|} h_k(I^r; 0) \exp(\mathrm{i}\langle k, \phi\rangle) \right\| \leqslant 4^{n+1} \frac{K_\sigma}{\kappa^n} \left(\frac{n-1}{\mathrm{e}}\right)^{n-1} \exp(-\tfrac{1}{2}\kappa |r_2|).$$

This estimate implies that if

$$|r_2| \geqslant L(\mu) = \mathrm{Int}\left(\frac{2}{\kappa} \log \frac{4^{n+1}(n-1)^{n-1} K_\sigma}{\mu\, \mathrm{e}^{n-1} \kappa^n}\right) + 1, \tag{6}$$

then the pendulum Hamiltonian (5) can be recast in the form

$$H_{\mathrm{pend}}(A, \alpha) = \tfrac{1}{2}\langle A, PA\rangle + V_1(\alpha_1; \mu) + \mu V_2(\alpha; \mu), \tag{7}$$

with

$$V_1(\alpha_1; \mu) = \sum_{p_1=0}^{L(\mu)/|r_1|} \tilde{h}_{(p_1, 0)} \exp(\mathrm{i} p_1 \alpha_1),$$

$$V_2(\alpha; \mu) = \sum_{|p_1 l| + |p_2 m| > L(\mu)} \frac{\tilde{h}_p}{\mu} \exp(\mathrm{i}\langle p, \alpha\rangle).$$

It is important to note that both V_1 and V_2 admit μ-independent bounds, in particular

$$|V_1| < K_\sigma, \quad |V_2| < 1. \tag{8}$$

Using Cauchy's inequality, we also obtain that $\|D_\phi^k V_2\| \leqslant |k|!\kappa^{-|k|}$ for any multi-index $k \in \mathbb{Z}^n$, hence the derivatives of the potential V_2 also admit μ-independent bounds, just as those of V_1. The remarkable feature of (7) is that it is a nearly integrable system for small values of μ. In order to study this system, we want to consider the $\mu = 0$ limit. To avoid the blowup of $L(\mu)$ that occurs at this limit, we introduce a new small parameter ρ with $0 \leqslant \rho \leqslant \mu$ and consider the slightly different Hamiltonian

$$H_{\mathrm{pend}}(A, \alpha; \rho) = \tfrac{1}{2}\langle A, PA\rangle + V_1(\alpha_1; \mu) + \rho V_2(\alpha; \mu). \tag{9}$$

Then, for any sufficiently small but fixed $0 < \mu \ll \min(1, K_\sigma)$, we can establish perturbation results for $H_{\mathrm{pend}}(A, \alpha; \rho)$ with $0 < \rho \leqslant \rho_0$ sufficiently small. If these perturbation results are obtained by methods that only require μ-independent bounds on the perturbation "potential" V_2 and its derivatives, than the results continue to hold if we decrease the value of μ to achieve $\mu = \rho_0$.

3. The integrable limit

The Hamiltonian vector field corresponding to $H_{\mathrm{pend}}(A, \alpha; 0)$ takes the form

$$\dot{A}_1 = -D_{\alpha_1} V_1(\alpha_1; \mu), \quad \dot{A}_2 = 0,$$
$$\dot{\alpha}_1 = p_{11} A_1 + p_{12} A_2, \quad \dot{\alpha}_2 = p_{22} A_2 + p_{12} A_2, \tag{10}$$

thus A_2 is an integral of this system. We assume that the periodic potential V_1 has isolated local minima and maxima and the nondegeneracy conditions

$$p_{11} \neq 0, \quad \det P \neq 0 \tag{11}$$

hold. Any local extremum point $\bar{\alpha}_1$ of V_1 gives rise to an equilibrium $(-p_{12}A_2/p_{11}, \bar{\alpha}_1)$ of the (A_1, α_1) equations, which yields a two-dimensional invariant manifold of (10) of the form

$$\mathcal{M}_0^{\bar{\alpha}_1} = \left\{ (A, \alpha) \mid \alpha_1 = \bar{\alpha}_1,\ A_1 = -\frac{p_{12}}{p_{11}} A_2 \right\}.$$

We are interested in the case when the equilibrium $(-p_{12}A_2/p_{11}, \bar{\alpha}_1)$ is a saddle, i.e., when the manifold $\mathcal{M}_0^{\bar{\alpha}_1}$ is normally hyperbolic. Such a manifold is either connected to itself or to some other hyperbolic invariant manifold by homoclinic or heteroclinic manifolds.

In Fig. 1 we show the geometry of the invariant manifold $\mathcal{M}_0^{\bar{\alpha}_1}$ by factoring out the angular coordinate α_2. We also indicate two important three-dimensional surfaces in the figure that appear as planes. The plane $A_1 = 0$ corresponds to the resonance hypersurface associated with the stronger r_1-resonance, while $A_2 = 0$ describes the hypersurface of the weaker r_2-resonance. We shall refer to these hypersurfaces as the *cores of the* respective *resonances*. It is simple to see that most solutions in $\mathcal{M}_0^{\bar{\alpha}_1}$ are periodic and, in terms of the full n-degree-of-freedom normal form (4), they correspond to the limits of $(n-1)$-dimensional whiskered tori

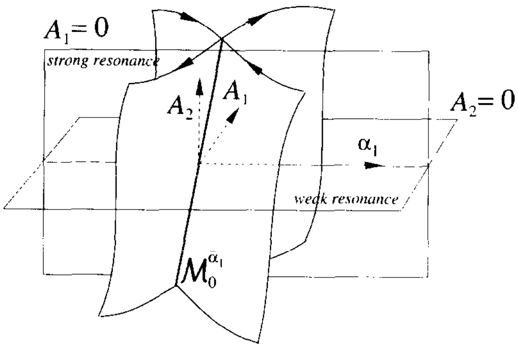

Fig. 1. The invariant manifold $\mathcal{M}_0^{\bar{\alpha}_1}$

that are created by the perturbation in the resonance junction. Note that the solutions of (10) have zero evolution in the direction of the weaker resonance; in particular, there are no solutions that cross the core of the weak resonance. At the same time, in any neighborhood of the manifold $\mathcal{M}_0^{\bar{\alpha}_1}$ most solutions do cross the core of the strong resonance.

Restricted to the manifold $\mathcal{M}_0^{\bar{\alpha}_1}$, the gradient of $H_{\text{pend}}(A, \alpha; 0)$ is of the form

$$DH_{\text{pend}}(A, \alpha; 0)|_{\mathcal{M}_0^{\bar{\alpha}_1}} = \left(0, 0, \frac{\det P}{p_{11}} A_2, 0\right).$$

The third entry in this vector is precisely the frequency of the periodic orbit in $\mathcal{M}_0^{\bar{\alpha}_1}$ which is labelled by the given constant value of A_2. By (11), this frequency is nonzero away from the core of the weak resonance ($A_2 = 0$). This implies that the three-dimensional energy surfaces of system (10) intersect $\mathcal{M}_0^{\bar{\alpha}_1}$ transversely along periodic orbits that are separated from the core of the weak resonance. (Transversality can be seen by noting that for any fixed $A_2 \neq 0$ and $\alpha_2 \in S^1$, the gradient DH_{pend} has a nonvanishing inner product with the vector $(-p_{12}/p_{11}, 0, 1, 0)$, which lies in the tangent space of is $\mathcal{M}_0^{\bar{\alpha}_1}$.) Hence, a more geometric explanation for the lack of motions crossing the weak resonance is the fact that the energy surfaces near $\mathcal{M}_0^{\bar{\alpha}_1}$ act as barriers to such motions.

The core of the weak resonance intersects the manifold $\mathcal{M}_0^{\bar{\alpha}_1}$ in an invariant circle (or *resonant circle*) C_r that satisfies $A_1 = A_2 = 0$. This circle does not carry periodic solutions, rather, it consists entirely of equilibria. This object has a great significance in the study of the resonance junction as it corresponds to the invariant $(n-1)$-dimensional torus of the integrable Hamiltonian H_0 that is located precisely at the center of the resonance junction, i.e., at the intersection of the cores of the weak and the strong resonances. In the limit $\rho = 0$ of the normal form, this torus is foliated by a one-parameter family of $(n-2)$-dimensional invariant tori which appear in system (10) as the equilibria contained in the resonant circle.

4. Diffusion across the weak resonance

The question we want to address now is how the flow near the circle C_r changes in the perturbed system

$$\dot{A}_1 = -D_{\alpha_1} V_1(\alpha_1; \mu) - \rho D_{\alpha_1} V_2(\alpha; \mu),$$
$$\dot{A}_2 = -\rho D_{\alpha_2} V_2(\alpha; \mu),$$
$$\dot{\alpha}_1 = p_{11} A_1 + p_{12} A_2, \quad \dot{\alpha}_2 = p_{12} A_1 + p_{22} A_2. \tag{12}$$

Basic invariant manifolds guarantee [14] that the manifold $\mathcal{M}_0^{\bar{\alpha}_1}$ will smoothly perturb into a $O(\rho)$ C^r-close invariant manifold $\mathcal{M}_\rho^{\bar{\alpha}_1}$ for any finite integer r. It is not difficult to show [15] that this perturbed manifold carries a one-degree-of-freedom Hamiltonian dynamics which slightly deforms but preserves the periodic solutions on $\mathcal{M}_\rho^{\bar{\alpha}_1}$ away from the $A_2 = 0$ core of the weak resonance. The perturbed energy-surfaces are now given by $H_{\text{pend}}(A, \alpha; \rho) = \text{const}$ and they keep intersecting the manifold $\mathcal{M}_\rho^{\bar{\alpha}_1}$ transversely with $O(1)$ transversality away from the set $A_2 = 0$. As a result, they remain barriers to motions in directions transverse to the weak resonance. However, such barriers are not guaranteed to survive in a neighborhood of the surface $A_2 = 0$ where the unperturbed energy surface is degenerate and has a nontransverse intersection with the manifold $\mathcal{M}_0^{\bar{\alpha}_1}$.

To understand what happens on $\mathcal{M}_\rho^{\bar{\alpha}_1}$ close to this degeneracy, we introduce the usual resonance scaling

$$A_2 = \sqrt{\rho}\, \eta,$$

which "blows up" a neighborhood of the core of the weak resonance. Then the Hamiltonian flow on $\mathcal{M}_\rho^{\bar{\alpha}_1}$ is generated by the restricted Hamiltonian $\mathcal{H}_\rho = H_{\text{pend}}|\mathcal{M}_\rho^{\bar{\alpha}_1}$, which is of the form

$$\mathcal{H}_\rho(\eta, \alpha_2) = \rho \mathcal{H}(\eta, \alpha_2) + O(\rho^{3/2}),$$
$$\mathcal{H}(\eta, \alpha_2) = \frac{1}{2} \frac{\det P}{p_{11}} \eta^2 + V_2(\bar{\alpha}_1, \alpha_2; \mu). \tag{13}$$

We refer to $\mathcal{H}$ as the *reduced Hamiltonian* and note that nonsingular level curves of this Hamiltonian smoothly approximate actual trajectories of the restricted Hamiltonian $\mathcal{H}_\rho$ with $O(\sqrt{\rho})$ precision (which means $O(\rho)$ precision in the original A_2 coordinate). These actual motions on $\mathcal{M}_\rho^{\bar{\alpha}_1}$ satisfy the equations

$$\dot{\eta} = -\sqrt{\rho}\, D_{\alpha_2}\mathcal{H}(\eta,\alpha_2) + O(\rho),$$

$$\dot{\alpha}_2 = \sqrt{\rho}\, D_{\eta}\mathcal{H}(\eta,\alpha_2) + O(\rho),$$

which are Hamiltonian, but in general are only canonical at leading order [15]. These equations show that $\mathcal{M}_\rho^{\bar{\alpha}_1}$ is locally a *slow manifold* that contains motions with a characteristic time scale of the order of $O(1/\sqrt{\rho})$. We call these slow motions *diffusion* on the perturbed manifold $\mathcal{M}_\rho^{\bar{\alpha}_1}$.

As seen from (13), the reduced Hamiltonian $\mathcal{H}$ is a one-degree-of-freedom potential-type Hamiltonian with mass $\det P/p_{11}$ and potential $V_2(\bar{\alpha}_1,\alpha_2;\mu)$. In general, by the nondegeneracy conditions (11), all equilibria of such a Hamiltonian lie on the α_2 axis and all level curves of $\mathcal{H}$ intersect the axis $\eta = 0$ transversely. A typical phase portrait for $\mathcal{H}$ is shown in Fig. 2a. Note that almost all orbits in an order $O(\sqrt{\rho})$ neighborhood of the core of the weak resonance *cross the core of the weak resonance* and connect points on opposite sides of this core that are $O(\sqrt{\rho})$ apart in their A_2 coordinates. The existence of these motions is due to the change in the topology of the energy surface $H_{\mathrm{pend}}(A,\alpha;\rho) = \mathrm{const}$ which removes the energy-barriers near $A_2 = 0$ and allows solutions to cross the weak resonance (see Fig 2b). (This new topology of the energy surfaces follows from the results of Fenichel on the invariant foliations of stable and unstable manifolds [10,11].) It is then plausible to define the *width of the weak resonance channel near* $\mathcal{M}_\rho^{\bar{\alpha}_1}$ by picking the maxima and minima of the separatrices that separate crossing and noncrossing trajectories on the slow manifold, and considering the strip lying between these two η values (see Fig. 2a). If $\bar{\alpha}_2$ is the angular coordinate of the hyperbolic equilibrium to which the two limiting separatrices asymptote, then the width of the weak resonance channel in terms of the original localized action variable A_2 is given by

$$\Delta A_2 = \sqrt{\rho}\Delta\eta,$$

$$\Delta\eta = \max_{\alpha_2\in S^1}\left(\frac{8p_{11}}{\det P}[V_2(\bar{\alpha}_1,\bar{\alpha}_2;\mu) - V_2(\bar{\alpha}_1,\alpha_2;\mu)]\right)^{1/2}. \tag{14}$$

We exclude a small but fixed neighborhood of the slow separatrices by picking a periodic orbit γ_U that bounds some elliptic region U (see Fig. 3a). We can then write down an upper bound on the time it takes for slow motions on $\mathcal{M}_\rho^{\bar{\alpha}_1}$ to diffuse from one side of the resonance channel to the other side. This upper bound is of the form

$$\Delta T(\mu) = \frac{\Delta\eta}{\sqrt{\rho}\, C_U}, \tag{15}$$

where $\sqrt{\rho}\, C_U$ is the modulus of the average velocity of a solution while it travels from one extremum point of the orbit γ_U to the other (as a result, the quantity $|C_U|$ is $O(1)$ as $\rho \to 0$).

We can also estimate the measure of initial conditions in the vicinity of the manifold $\mathcal{M}_\rho^{\bar{\alpha}_1}$ for which the corresponding solutions exhibit a similar $O(\sqrt{\rho})$ diffusion through the core of the weak resonance. First, we define the open annular region $V \subset U$ as shown in Fig 3a. Note that points in V are initial conditions for solutions that connect points on different sides of the weak resonance with action η coordinates at least $\frac{1}{4}\Delta\eta$ apart from $\eta = 0$. We define the constant $c_1 > 0$ so that on each solution in the set V the time of passage from one extremum point to the other on the opposite side of the weak resonance is bounded by $c_1/\sqrt{\rho}$.

The key tool we use at this point is a general normal form for (12) near hyperbolic slow manifolds which is originally due to Fenichel [10,16]. Near the slow part of the manifold $\mathcal{M}_\rho^{\bar{\alpha}_1}$ and for $(A_2/\sqrt{\rho},\alpha_2) \in U$, this normal form is given by

$$\begin{aligned}
\dot{x}_1 &= [-\lambda + X_1(x,y)x_1 + X_2(x,y)x_2 \\
&\quad + \sqrt{\rho}\, X_3(x,y)]x_1, \\
\dot{x}_2 &= [\lambda + X_4(x,y)x_1 + X_5(x,y)x_2 \\
&\quad + \sqrt{\rho}\, X_6(x,y)]x_2, \\
\dot{y}_1 &= \sqrt{\rho}\, Y_1(x,y)x_1x_2, \\
\dot{y}_2 &= \sqrt{\rho}\, Y_2(y_1) + Y_3(x,y)x_1x_2.
\end{aligned} \tag{16}$$

Here the functions X_i and Y_j are as smooth as needed and they also depend on the parameter $\sqrt{\rho}$. The pos-

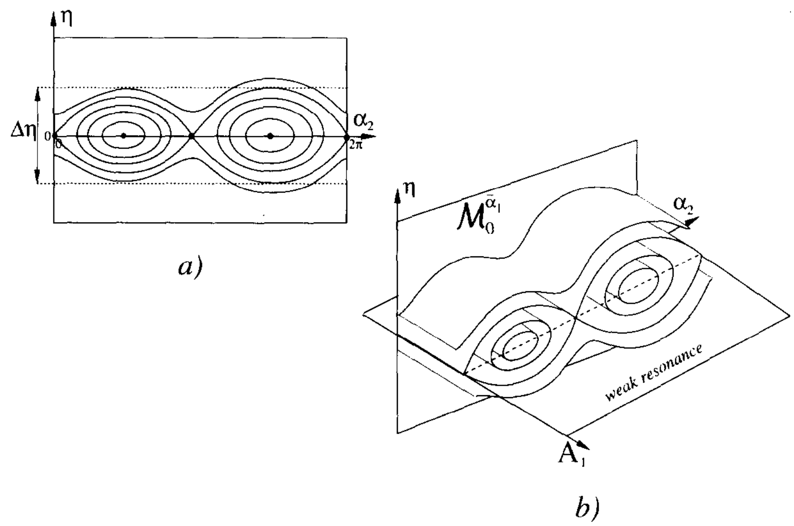

Fig. 2. (a) Phase portrait for the reduced Hamiltonian. (b) Schematic picture of energy surfaces near the weak resonance for $\rho > 0$.

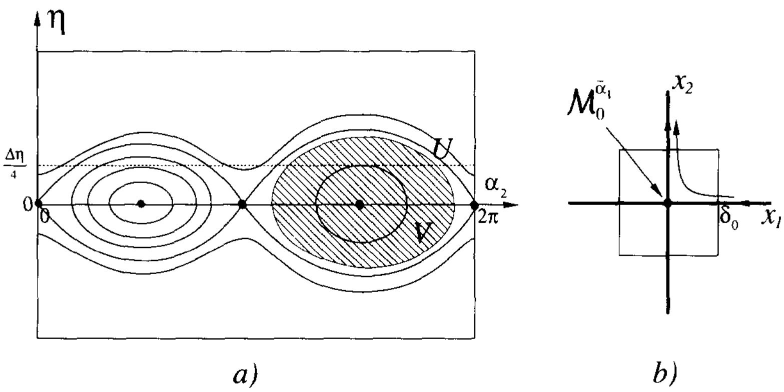

Fig. 3. (a) Definition of the sets U and V. (b) Local geometry near the slow manifold.

itive constant λ is equal to the positive eigenvalue of the Jacobian $D^2 H_{\text{pend}}(0,0,\bar{\alpha}_1;0)$. The slow manifold $\mathcal{M}_\rho^{\bar{\alpha}_1}$ is now simply given by $x=0$ and its local stable and unstable manifolds are given by $x_2=0$ and $x_1=0$, respectively. The coordinates (y_1,y_2) are action-angle-type variables for the periodic orbits in the domain U. We want to determine the possible initial conditions for a solution $(x(t),y(t))$ that enters a small box of size $2\delta_0$ around the slow manifold (see Fig. 3b) and stays close for times $t \leqslant \Delta T(\mu)$ to a slow solution $(x_s(t),y_s(t))$ in the set V. A simple estimate based on the Fenichel norm form (16) shows that the x_2 coordinates of such trajectories at entry have to satisfy $|x_2(0)| < \delta_0 \exp(-3c_1\lambda/4\sqrt{\rho})$, otherwise these trajectories would leave the δ_0-box before the time $\Delta T(\mu)$. (Here $2c_1/\sqrt{\rho}$ is a lower bound on the periods of the solutions inside V.) In that case, (16) yields the estimate $|x_1(t)x_2(t)| < K\delta_0^2 \exp(-c_1\lambda/2\sqrt{\rho})$ for all $t \in [0,\Delta T(\mu)]$ and for δ_0 sufficiently small, which in turn implies

$$\sup_{t\in[0,\Delta T(\mu)]} |y_1(t) - y_{s1}(t)| < K_2\delta_0^2 \exp(-c_1\lambda/2\sqrt{\rho}),$$

$$\sup_{t\in[0,\Delta T(\mu)]} |y_2(t) - y_{s2}(t)| < K_3\delta_0^2 \exp(-c_1\lambda/2\sqrt{\rho}).$$

Thus the y-coordinates of the solution passing through the δ_0-box indeed stays (exponentially) close to the y-coordinates of the slow solution. As a result, we can select an open set $\mathcal{I}$ of initial conditions around the slow manifold with measure

$$\begin{aligned} &\mathrm{mes}_A(\mathcal{I}) \\ &\quad > K_4[\mathrm{Area}(V) \times \delta_0 \times \delta_0 \exp(-c_1\lambda/2\sqrt{\rho})] \\ &\quad = \mathrm{O}(\exp(-c_1\lambda/2\sqrt{\rho})), \end{aligned} \tag{17}$$

such that solutions starting from the set $\mathcal{I}$ *diffuse through the weak resonance along the strong resonance.* (Here $\mathrm{mes}_A(\ )$ denotes the Lebesgue measure in the (A,α)-space.) If the estimates (15) and (17) hold for $0 < \rho \leqslant \rho_0$, then decreasing the value of μ to ρ_0 and using (6) we obtain the following diffusion time and diffusion measure estimates for motions of the original Hamiltonian (1),

$$\begin{aligned} &\Delta T < \frac{1}{\sqrt{\epsilon}} \frac{C_1(n)\kappa^{n/2}\exp(\frac{1}{4}\kappa|r_2|)}{\sqrt{K_\sigma}}, \\ &\mathrm{mes}_I(\mathcal{I}) > \epsilon^{n/2} C_2 \exp\left(-\frac{\lambda\kappa^n \exp(\frac{1}{2}\kappa|r_2|)}{C_3(n)\sqrt{K_\sigma}}\right). \end{aligned} \tag{18}$$

Here $C_2 > 0$, and the positive constants $C_1(n)$ and $C_2(n)$ depend only on the number of degrees-of-freedom. The factor $1/\sqrt{\epsilon}$ in the diffusion time estimate is the result of the $\sqrt{\epsilon}$ factor multiplying the pendulum Hamiltonian in (4), and the $\epsilon^{n/2}$ factor in the measure estimate enters because of the transformation (3). Note that while the lower estimate for $\mathrm{mes}_I(\mathcal{I})$ is super-exponentially small in terms of the order of the weaker resonance, it is *algebraic in the perturbation parameter* ϵ. This fact is quite remarkable because if one performs similar estimates for the measure of initial conditions that exhibit Arnold diffusion along a single resonance of Hamiltonian (1), one obtains a measure that is exponentially small in ϵ [6]. To summarize, we can conclude that while diffusion across weak–strong resonance junctions of the Hamiltonian (1) is necessarily restricted to lengths of the order of $\mathrm{O}(\sqrt{\epsilon})$ as $\epsilon \to 0$, its speed and the measure of initial conditions it effects is much larger than the same quantities for Arnold diffusion along a single resonance, even when the latter are computed for similar lengths in the action space.

5. Conclusions

In this note we have derived a normal form to describe motions near the intersection of a weaker and a stronger resonance in an n-degree-of-freedom, nearly integrable Hamiltonian system of form (1). Using the exponential decay in the Fourier series of the analytic perturbation, we have shown that if the order of the second resonance obeys estimate (6) for some small number $\mu \ll \|H_1(I^r,\phi;\epsilon)\|$, then the normal form is an order $\mathrm{O}(\mu)$ perturbation of an integrable pendulum-type equation that depends only on the stronger resonant combination of the phases. As a result of its logarithmic dependence on $1/\mu$, condition (6) does not require a very high order $|r_2|$ for the second resonance, thus we expect our results to be relevant for a large class of resonance junctions.

We have analyzed the integrable part of the normal form and identified energetical barriers that prevent motions from crossing the weaker resonance along the stronger one. However, near an $(n-1)$-dimensional, doubly-resonant torus at the center of the junction, these barriers are singular and are typically destroyed by the perturbative effect of the weaker resonance. Indeed, in terms of the pendulum-type Hamiltonian, the singularity perturbs into a normally hyperbolic, invariant slow manifold that contains periodic solutions that cross the weaker resonance. These slow periodic solutions correspond to $(n-1)$-tori in the full truncated normal form which possess one slow phase. The survival of these tori under the exponentially small tail of the normal form is a more subtle question which we did not address here [11]. However, the existence of solutions that connect the opposite sides of the weak resonance while moving along the strong resonance is independent of the persistence of the above tori. The reason is that for ϵ sufficiently small, $1/\sqrt{\epsilon} \ll \mathrm{e}^{c/\epsilon}$ (see (18)), thus, by basic results from multi-phase averaging, the above motions do approximate actual near-resonance motions of Hamiltonian (1) with exponentially small error on the time scale of the estimated crossing time ΔT [9,13,11]. We also gave a lower estimate in (18) for the measure of initial conditions that exhibit the same cross-resonance diffusion. Our estimates are *algebraic* in the perturbation parameter ϵ as opposed to similar estimates for Arnold diffusion along a single resonance, which yield a lower bound that is exponentially small in ϵ. This fact gives an explanation for the numerical results in Ref. [8] which indicate that cross-resonance diffusion appears to dominate Arnold diffusion and it frequently involves higher order resonances.

The diffusing trajectories we constructed in this note are regular motions that may cross the weak resonance repeatedly. In a companion paper [11] we show the existence of complicated, multi-pulse homoclinic orbits that asymptote to the crossing motions in forward and backward time after an intermediate "jumping" along the stronger resonance. These homoclinic orbits turn out to admit a universal bifurcation diagram that can be described by an infinite binary tree. Furthermore, these orbits are created via $O(\sqrt{\epsilon})$ *splitting of separatrix surfaces* which results in more intense chaotic dynamics than in the case of diffusion along a single resonance.

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