



## Addendum to ‘Coherent Lagrangian vortices: the black holes of turbulence’

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In Haller & Beron-Vera (*J. Fluid Mech.*, vol. 731, 2013, R4) we developed a variational principle for the detection of coherent Lagrangian vortex boundaries. The solutions of this variational principle turn out to be closed null geodesics of the Lorentzian metric induced by a generalized Green–Lagrange strain tensor family. This metric interpretation implies a mathematical analogy between coherent Lagrangian vortex boundaries and photon spheres in general relativity. Here, we give an improved discussion of this analogy.

**Key words:** geophysical and astrophysical flows, geostrophic turbulence, ocean circulation

### 1. The main results of Haller & Beron-Vera (2013)

We consider a two-dimensional velocity field  $v(x, t)$ , with  $x$  labelling the location within a two-dimensional region  $U$  of interest and with  $t$  referring to time. Fluid trajectories generated by  $v(x, t)$  are denoted  $x(t; t_0, x_0)$ , with  $x_0$  referring to the initial position of the trajectory at time  $t_0$ . These trajectories solve the differential equation

$$\dot{x} = v(x, t) \quad (1.1)$$

and generate the flow map

$$F_{t_0}^t(x_0) := x(t; t_0, x_0), \quad (1.2)$$

which takes an initial position  $x_0$  at time  $t_0$  to its current position at time  $t$ .

The right Cauchy–Green strain tensor field associated with the flow map is defined as  $C_{t_0}^t(x_0) = \nabla F_{t_0}^t(x_0)^\top \nabla F_{t_0}^t(x_0)$ , with eigenvalues  $\lambda_i(x_0)$  and eigenvectors  $\xi_i(x_0)$  satisfying

$$C_{t_0}^t \xi_i = \lambda_i \xi_i, \quad |\xi_i| = 1, \quad i = 1, 2; \quad 0 < \lambda_1 \leq \lambda_2, \quad \xi_1 \perp \xi_2. \quad (1.3)$$

In Haller & Beron-Vera (2013) we sought the time  $t_0$  positions of Lagrangian vortex boundaries as closed stationary curves of the averaged Lagrangian strain. Such curves

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turned out to coincide with the zero-energy solutions of a one-parameter family of variational problems defined as

$$\delta\mathcal{E}_\lambda(\gamma) = 0, \quad \mathcal{E}_\lambda(\gamma) = \oint_\gamma \langle r'(s), E_\lambda(r(s))r'(s) \rangle ds, \quad \lambda \in \mathbb{R}^+. \quad (1.4)$$

Here, the strain energy functional  $\mathcal{E}_\lambda(\gamma)$  is defined through the generalized Green–Lagrange strain tensor family

$$E_\lambda(x_0) = \frac{1}{2}[\mathbf{C}'_{t_0}(x_0) - \lambda^2 \mathbf{I}], \quad \text{where } \mathbf{I} \text{ is the identity matrix.} \quad (1.5)$$

Consider the flow domain

$$U_\lambda = \{x_0 \in U \mid \lambda_1(x_0) < \lambda^2 < \lambda_2(x_0)\}, \quad (1.6)$$

where the tensor field  $E_\lambda$  has two non-zero eigenvalues of opposite sign. Then the quadratic function

$$g_\lambda(u, u) = \langle u, E_\lambda u \rangle \quad (1.7)$$

defines a Lorentzian metric (Beem, Ehrlich & Kevin 1996) on  $U_\lambda$ , with signature  $(-, +)$  inherited from the eigenvalue configuration of  $E_\lambda$ . The zero-energy solutions of (1.4) are therefore precisely the closed null geodesics of the Lorentzian metric  $g_\lambda$ , which satisfy one of the two differential equations

$$r'(s) = \eta_\lambda^\pm(r(s)), \quad \eta_\lambda^\pm(r) = \sqrt{\frac{\lambda_2(r) - \lambda^2}{\lambda_2(r) - \lambda_1(r)}} \xi_1(r) \pm \sqrt{\frac{\lambda^2 - \lambda_1(r)}{\lambda_2(r) - \lambda_1(r)}} \xi_2(r). \quad (1.8a,b)$$

In Haller & Beron-Vera (2013) we concluded that closed orbits of (1.8) (termed closed  $\lambda$  lines) must necessarily encircle metric singularities of  $g_\lambda$ . Such singularities occur at points  $x_0$  where  $\lambda_1(r) = \lambda_2(r)$  holds for the eigenvalues of the Cauchy–Green strain tensor.

The Lorentzian metric interpretation discussed above implies a geometric analogy between coherent Lagrangian vortex boundaries and photon spheres in cosmology. Below, we give more detail on this analogy, followed by an improved version of its summary with a more relevant reference.

## 2. More on the analogy with photon spheres

In the vicinity of any  $\lambda$  line in  $U_\lambda$ , the vector fields  $\xi_i$  define a curvilinear coordinate system with pointwise orthogonal coordinate lines. Direct substitution of  $\xi_i$  into the metric (1.7) gives

$$g_\lambda(\xi_i, \xi_i) = \frac{1}{2}(\lambda_i - \lambda^2), \quad (2.1)$$

showing that  $g_\lambda(\xi_1, \xi_1) < 0$  and  $g_\lambda(\xi_2, \xi_2) > 0$  everywhere in  $U_\lambda$ . This shows that the  $\xi_1$  trajectories form the time-like coordinates and the  $\xi_2$  trajectories form the space-like coordinates of the metric  $g_\lambda$  in  $U_\lambda$  (Beem *et al.* 1996). We further note that  $g_\lambda(\eta_\lambda^\pm, \eta_\lambda^\pm) = 0$ , and hence a  $\lambda$  line is nowhere space-like in the language of Lorentzian geometry. Given that our closed null geodesics are nowhere space-like hypersurfaces built out of null geodesics, they are photon surfaces by the general definition of Claudel, Virbhadra & Ellis (2001).

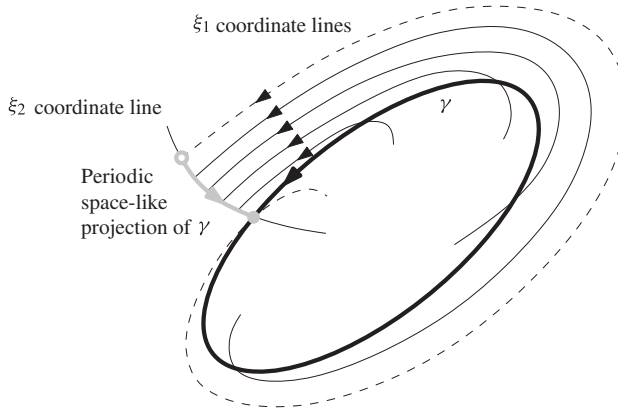


FIGURE 1. A closed null geodesic  $\gamma$  of the Lorentzian metric  $g_\gamma$  has a periodic space-like projection along the time-like coordinate lines. This space-like projection has a jump discontinuity due to the low dimensionality of the space–time, but evolves periodically along with the underlying closed null geodesic. This is in contrast to general  $\lambda$  lines which have aperiodic space-like projections.

Next, we note that

$$\langle \eta_\lambda^\pm(r), \xi_1(r) \rangle = \sqrt{\frac{\lambda_2(r) - \lambda^2}{\lambda_2(r) - \lambda_1(r)}} \in (0, 1), \quad r \in U_\lambda. \quad (2.2)$$

Consequently, trajectories of the  $\xi_1(r)$  line field ( $\xi_1$  coordinate lines) intersect any closed  $\lambda$  line  $\gamma$  transversely, with an angle of intersection that is always less than  $\pi/2$ , as sketched qualitatively in figure 1. In the same figure, we also show a representative trajectory of the  $\xi_2(r)$  line field (a  $\xi_2$  coordinate line), which is pointwise orthogonal to the  $\xi_1$  coordinate lines by construction. We conclude that the projection of  $\gamma$  onto a space-like submanifold along the time-like coordinates results in a periodic (albeit discontinuous) space-like orbit (cf. figure 1).

In summary, a closed orbit  $\gamma$  of the  $\eta_\lambda^\pm(r)$  vector field is a photon surface of the  $(U_\lambda, g_\lambda)$  space–time. Geodesics forming this photon surface have periodically moving projections on the space-like coordinates, with the projection taken along the time-like coordinate lines.

The orbit  $\gamma$ , therefore, satisfies a plausible extension of the Claudel–Virbhadra–Ellis definition of a photon sphere (Claudel *et al.* 2001) from symmetric higher-dimensional space–times to non-symmetric two-dimensional space–times. This extension relaxes the requirement for a rotational symmetry and smoothness of the projected spatially periodic orbits. Both of these features are unattainable in curved two-dimensional space–times, and hence can reasonably be waived.

### 3. Revised wording of the cosmological analogy

In view of the above discussion, the wording of the mathematical analogy between coherent Lagrangian eddy boundaries and photon spheres in cosmology (Haller & Beron-Vera 2013, p. 731, para. 2) should be revised as follows.

The closed  $\lambda$  lines we have been seeking are therefore closed null geodesics of  $g_\lambda$ . Trajectories spanning these geodesics project along the

local time-like coordinates (trajectories of the  $\xi_1$  vector field) onto periodic trajectories on the local space-like coordinates (trajectories of the  $\xi_2$  vector field). In cosmology, the projection of some families of null geodesics from space-time onto the space-like variables also produces closed trajectories. Such families of null geodesics are often referred to as photon spheres, as their spatial projections trap photons orbiting around black holes (Claudel *et al.* 2001). In the cosmological context, null geodesics are tangent to light cones, which in our case are formed by the two vectors  $\eta_\lambda^\pm(x_0)$  at  $x_0$  (figure 2 of Haller & Beron-Vera 2013).

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