Unsteady fluid flow separation by the method of averaging

M. S. Kilic
Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139

G. Haller
Department of Mechanical Engineering, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139

A. Neishtadt
Space Research Institute, Russian Academy of Sciences, Profsoyuznaya Street 84/32, 117810 Moscow, Russia

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We use the method of averaging to improve recent separation criteria for two-dimensional unsteady fluid flows with no-slip boundaries. Our results apply to general compressible flows that admit a well-defined asymptotic average. Such flows include periodic and quasiperiodic flows, as well as aperiodic flows with a mean component. As an example, we predict and verify the unsteady separation location and angle in variants of an oscillating separation bubble model. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1924698]

I. INTRODUCTION

Fluid flow separation is generally regarded as the detachment of fluid from a no-slip boundary. Expressed in terms of energy principles, separation takes place when the kinetic energy of the flow near the wall is depleted by the viscous stresses within the boundary layer. These large energy losses typically lead to a degradation in the operational performance of engineering devices. For example, separation on a bluff body, such as a circular cylinder, increases the pressure drag dramatically, whereas in a diffuser, separation decreases the pressure recovery.

A. Prior work on flow separation

In a classic paper, Prandtl[1] derived a separation criterion for two-dimensional steady incompressible velocity fields $\mathbf{v}(\mathbf{x})=[u(x,y),v(x,y)]$ that satisfy the no-slip boundary condition $u(x,0)=v(x,0)=0$ on the $y=0$ boundary. Prandtl's criterion says that separation takes place at a boundary point $\mathbf{x}_0=(\gamma,0)$ whenever

$$u_y(\gamma,0)=0,$$

$$u_{xy}(\gamma,0)<0. \tag{1}$$

In physical terms, Prandtl's steady separation criterion requires zero wall-shear and negative wall-shear gradient. Under these conditions, the fluid breaks away from the boundary along a streamline that emanates from $\mathbf{x}_0$, the separation point. In dynamical systems terms, the separating streamline is the unstable manifold of $\mathbf{x}_0$ in the particle-motion equation of $\dot{\mathbf{x}}=\mathbf{v}(\mathbf{x})$. Due to the no-slip boundary conditions, this unstable manifold is nonhyperbolic: the velocity gradient $\nabla\mathbf{v}$ admits a pair of zero eigenvalues at $\mathbf{x}_0$.

For time-dependent velocity fields of the form

$$\mathbf{v}(\mathbf{x},t)=[u(x,y,t),v(x,y,t)], \tag{2}$$

with the no-slip boundary condition

$$u(x,0,t)=v(x,0,t)=0, \tag{3}$$

the steady separation criterion (1) fails, as was already noted by Sears and Tellionis.[2] Specifically, instantaneous zeros of the wall shear do not coincide with the locations of fluid breakaway. A number of authors, most notably Sears and Tellionis[2] and Van Dommelen and Shen,[3] proposed extensions of Prandtl's criterion to unsteady flows, but a generally applicable and rigorous criterion did not emerge (see the work by Haller[4] for a survey).

Studying the time-periodic incompressible flows, Shariff, Pulliam, and Ottino[5] realized that *fixed separation* (i.e., separation at a constant location) in unsteady flows can still be viewed as a material ejection from the boundary along a time-dependent unstable manifold, i.e., a material line that shrinks to the separation point in backward time (see Fig. 1). By contrast, *moving separation* (i.e., separation at time-varying locations) cannot be described by classical unstable manifolds, because that would contradict the invariance of those manifolds.

Shariff, Pulliam, and Ottino[5] argued that a necessary condition for fixed separation in two-dimensional time-periodic incompressible flows is

$$\int_0^T u_y(\gamma,0,t)dt=0, \tag{4}$$

with T denoting the period. This zero mean-skin-friction principle, however, was obtained from an unverified assumption on the associated Poincaré map, as was pointed out by Yuster and Hackborn.[6]

The latter authors rederived (4) rigorously for small time-periodic perturbations of steady incompressible flows. For such flows, Yuster and Hackborn[6] also found that condi-

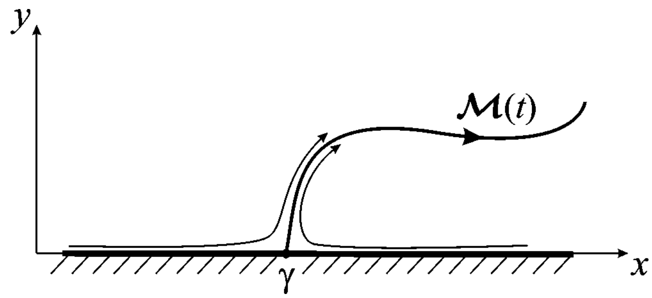

FIG. 1. Fluid separation along a nonhyperbolic time-dependent unstable manifold $\mathcal{M}(t)$.

tion (4), as well as the second condition in (1) applied to the steady limit, give a sufficient separation criterion. Their approach, however, does not extend to aperiodic, compressible, or far-from-steady flows.

Recently, Haller[4] showed that in any two-dimensional velocity field, fixed separation points satisfy

$$\limsup_{t\to-\infty}\left|\int_{t_0}^{t}\exp\left[\int_{t_0}^{\tau}v_y(\gamma,0,s)ds\right]u_y(\gamma,0,\tau)d\tau\right|<\infty, \tag{5}$$

where t_0 is an arbitrary but fixed time. In mathematical terms, (5) is a necessary condition for a boundary point $(\gamma,0)$ to admit a nonhyperbolic unstable manifold that remains uniformly bounded away from the $y=0$ boundary for all times $t\leqslant t_0$. For steady flows, (5) becomes identical to Prandtl's first separation condition in (1).

For incompressible flows, Haller[4] showed that a sufficient condition for a fixed unsteady separation at $(\gamma,0)$ is (5) appended with

$$u_{xy}(\gamma,0,t)<-c_0<0,\quad t\in\mathbb{R}, \tag{6}$$

where $c_0>0$ is a constant. Again, for steady flows, (6) simplifies to Prandtl's second condition in (1). Haller[4] also formulated a theory of moving unsteady separation using the concept of *finite-time unstable manifolds*.

B. Results

In this paper, we improve the sufficient separation criteria (5) and (6) for two-dimensional unsteady velocity fields that admit a finite asymptotic average in time. For such flows, we show that a sufficient criterion for fixed unsteady separation at $(\gamma,0)$ is

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\exp\left[\int_{t_0}^{t}v_y(\gamma,0,s)ds\right]u_y(\gamma,0,t)dt=0,$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\exp\left[\int_{t_0}^{t}v_y(\gamma,0,s)ds\right]\left[u_{xy}(\gamma,0,t)+u_y(\gamma,0,t)\int_{t_0}^{t}v_{xy}(\gamma,0,s)ds\right]dt<0,$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\exp\left[\int_{t_0}^{t}v_y(\gamma,0,s)ds\right]v_{yy}(\gamma,0,t)dt>0, \tag{7}$$

where t_0 is arbitrary but fixed. The first of the above three conditions is a necessary separation criterion that follows directly from (5). The separation criterion (7) also applies to moving and curved no-slip boundaries after a simple change of coordinates, as explained by Haller.[4]

For velocity fields with a well-defined asymptotic mean, the sufficient separation condition (7) improves (5) and (6) in two ways. First, (7) covers general compressible flows, and hence applies to real-life aerodynamic problems. Second, (7) only requires the weighted average of the skin-friction gradient to be negative, as opposed to (6), which requires the skin-friction gradient to be negative for all times.

For incompressible flows, we have $u_{xy}(\gamma,0)=-v_{yy}(\gamma,0)$, and the no-slip boundary condition implies $u_x(x,0,t)=v_y(\gamma,0,t)=0$. Thus, (7) simplifies to

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}u_y(\gamma,0,t)dt=0,$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}u_{xy}(\gamma,0,t)dt<0. \tag{8}$$

For steady incompressible flows, (8) further simplifies to the Prandtl condition (1).

We derive (7) by transforming the general compressible velocity field (2) to a normal form which is a small perturbation of a steady incompressible flow in the vicinity of the boundary. We then combine the method of aperiodic averaging with a topological invariant manifold construction to show the existence of a nonhyperbolic unstable manifold emanating from the boundary point $(\gamma,0)$ whenever (7) holds. Next, we derive the time-varying slope of this manifold from second-order averaging. Finally, we illustrate our results on periodic, quasiperiodic, and aperiodic versions of a two-dimensional separation bubble flow originally derived by Ghosh, Leonard, and Wiggins.[7]

II. NORMAL FORM NEAR A NO-SLIP WALL

A. Assumptions

We assume that the velocity field $\mathbf{v}(\mathbf{x},t)$ in (2) satisfies the no-slip boundary condition (3). We further assume that the fluid conserves mass near the separation point, i.e., the fluid density field $\rho(\mathbf{x},t)$ locally satisfies the continuity equation

$$\rho_t+\nabla\cdot(\rho\mathbf{v})=0. \tag{9}$$

Because of the no-slip boundary conditions (3), Eq. (9) simplifies to

$$\rho_t(x,0,t)=\rho(x,0,t)v_y(x,0,t)$$

along the boundary, yielding the relation

$$\rho(x,0,t)=\rho(x,0,t_0)\exp-\left[\int_{t_0}^{t}v_y(x,0,s)ds\right].$$

We further assume that $\rho(x,0,t)$ remains uniformly bounded along the boundary, which implies

$$\left| \int_{t_0}^{t} v_y(x,0,s)ds \right| \leq K_d < \infty \tag{10}$$

for some positive constant K_d. Note that for incompressible flows, $v_y(x,0,t)=-u_x(x,0,t)\equiv 0$, and hence (10) is always satisfied.

We also assume that for any finite time t_0, the velocity field $\mathbf{v}$ admits a finite asymptotic average

$$\mathbf{v}^0(\mathbf{x}) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0-T}^{t_0} \mathbf{v}(\mathbf{x},t)dt \tag{11}$$

on bounded sets. In their discussion on aperiodic averaging, Sanders and Verhulst[8] call such a $\mathbf{v}(\mathbf{x},t)$ a Krylov–Bogoliubov–Mitropolsky vector field.

Finally, we assume that the function

$$\mathbf{\Delta}(\mathbf{x},t) = \int_{t_0}^{t} [\mathbf{v}(\mathbf{x},s) - \mathbf{v}^0(\mathbf{x})]ds, \tag{12}$$

as well as its spatial derivatives up to third order, remain bounded uniformly for $t \leq t_0$ on bounded sets.

B. Locally incompressible normal form

Using the no-slip boundary conditions (3), we rewrite $\mathbf{v}$ as

$$\mathbf{v}(\mathbf{x},t) = [yA_1(x,y,t), yB_1(x,y,t)],$$

with the functions

$$A_1(x,y,t) = \int_0^1 u_y(x,sy,t)ds,$$

$$B_1(x,y,t) = \int_0^1 v_y(x,sy,t)ds.$$

Fluid particle motions then satisfy the differential equation

$$\dot{x} = yA_1(x,y,t),$$

$$\dot{y} = yB_1(x,y,t). \tag{13}$$

If $\mathbf{v}$ is incompressible, then $B_1(x,0,t)=0$, and hence the second equation in (13) contains no linear term in y.

In compressible flows, the linear y term in $yB_1(x,y,t)$ can still be removed by letting

$$y = \exp\left[\int_{t_0}^{t} B_1(x,0,s)ds\right]\tilde{y} = \exp\left[\int_{t_0}^{t} v_y(x,0,s)ds\right]\tilde{y}, \tag{14}$$

where t_0 is an arbitrary fixed time. Indeed, substituting (14) into (13) and dropping the tilde, we obtain the normal form

$$\dot{x} = yA_2(x,y,t),$$

$$\dot{y} = y^2C_2(x,y,t), \tag{15}$$

for particle motions near the boundary, with

$$A_2(x,y,t) = \exp\left[\int_{t_0}^{t} v_y(x,0,s)ds\right] A_1\left(x, y\exp\int_{t_0}^{t} v_y(x,0,s)ds, t\right),$$

$$C_2(x,y,t) = \exp\left[\int_{t_0}^{t} v_y(x,0,s)ds\right] D_yB_1(x,0,t) + \mathcal{O}(y).$$

C. Near-steady locally incompressible normal form

To focus on the dynamics near the $y=0$ boundary, we apply the rescaling $y\to\epsilon y$ in (15) with a small positive parameter ϵ to obtain

$$\dot{\mathbf{x}} = \epsilon\mathbf{f}(\mathbf{x},t) + \epsilon^2\mathbf{g}(\mathbf{x},t;\epsilon), \tag{16}$$

where

$$\mathbf{f}(\mathbf{x},t) = \begin{pmatrix} yA_2(x,0,t) \\ y^2C_2(x,0,t) \end{pmatrix},$$

$$\mathbf{g}(\mathbf{x},t;\epsilon) = \begin{pmatrix} y^2[D_yA_2(x,0,t) + \mathcal{O}(\epsilon y)] \\ y^3[D_yC_2(x,0,t) + \mathcal{O}(\epsilon y)] \end{pmatrix}.$$

For small ϵ, (16) is a slowly varying system. We, therefore, expect the main features of the particle dynamics near the $y=0$ boundary to be captured by the averaged version of (16). Indeed, as we show in Appendix A, there exists a change of variables $\mathbf{x}=(x,y)\mapsto\boldsymbol{\xi}=(\xi,\eta)$ under which (16) becomes

$$\dot{\boldsymbol{\xi}} = \epsilon\mathbf{f}^0(\boldsymbol{\xi}) + \epsilon^2\mathbf{f}^1(\boldsymbol{\xi},t) + \mathcal{O}(\epsilon^3), \tag{17}$$

with

$$\mathbf{f}^0(\boldsymbol{\xi}) = \begin{pmatrix} \eta\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0} A_2(\xi,0,t)dt \\ \eta^2\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0} C_2(\xi,0,t)dt \end{pmatrix},$$

$$\mathbf{f}^1(\boldsymbol{\xi},t,\epsilon) = D_{\boldsymbol{\xi}}\mathbf{f}(\boldsymbol{\xi},t)\mathbf{w} - D_{\boldsymbol{\xi}}\mathbf{w}\mathbf{f}^0(\boldsymbol{\xi}) + \mathbf{g}(\boldsymbol{\xi},t;0). \tag{18}$$

This result can also be derived from the classic work of Bogoliubov and Mitropolsky[9] on asymptotic averaging.

III. SUFFICIENT CONDITION FOR FIXED UNSTEADY SEPARATION

In view of the normal form (17), compressible flows near a no-slip boundary can be viewed as small perturbations of steady incompressible flows. After the change of coordinates $\mathbf{x}\mapsto\boldsymbol{\xi}$, therefore, we can construct the time-dependent separation profiles as small perturbations of the steady separation profiles. Such steady profiles can be located by applying Prandtl's criterion to the leading-order steady velocity

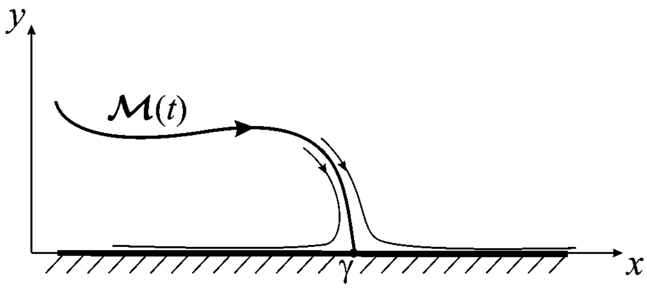

FIG. 2. Fuid reattachment along a nonhyperbolic time-dependent stable manifold $\mathcal{M}(t)$.

field in (17). This approach yields the following sufficient condition for unsteady separation in flows with an asymptotic mean.

Theorem 1. Suppose that the assumptions of Sec. III A hold, and

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\exp\left[\int_{t_0}^{t}v_y(\gamma,0,s)ds\right]u_y(\gamma,0,t)dt=0,$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\exp\left(\int_{t_0}^{t}v_y(\gamma,0,s)ds\right)\left[u_{xy}(\gamma,0,t)+u_y(\gamma,0,t)\int_{t_0}^{t}v_{xy}(\gamma,0,s)ds\right]dt<0,$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\exp\left[\int_{t_0}^{t}v_y(\gamma,0,s)ds\right]v_{yy}(\gamma,0,t)dt>0 \qquad (19)$$

are satisfied for some time t_0 at a boundary point $\mathbf{p}=(\gamma,0)$. Then $\mathbf{p}$ is a fixed unsteady separation point, i.e., $\mathbf{p}$ admits a nonhyperbolic unstable manifold that remains uniformly bounded away from the $y=0$ boundary for all $t\leq t_0$.

We prove Theorem 1 in Appendix B.

As noted by Haller,[4] *fixed unsteady reattachment* can be viewed as the convergence of fluid particles to a no-slip boundary along a nonhyperbolic *stable manifold* (see Fig. 2). Reversing time in the proof of Theorem 1, we obtain that fixed unsteady reattachment points satisfy

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\exp\left[\int_{t_0}^{t}v_y(\gamma,0,s)ds\right]u_y(\gamma,0,t)dt=0,$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\exp\left(\int_{t_0}^{t}v_y(\gamma,0,s)ds\right)\left[u_{xy}(\gamma,0,t)+u_y(\gamma,0,t)\int_{t_0}^{t}v_{xy}(\gamma,0,s)ds\right]dt>0,$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\exp\left[\int_{t_0}^{t}v_y(\gamma,0,s)ds\right]v_{yy}(\gamma,0,t)dt<0.$$

IV. SEPARATION PROFILE FROM HIGHER-ORDER AVERAGING

By the structure of the averaged normal form (17), we obtain $(x,y)=(\gamma,y)$ as a first-order approximation for the separation profile, i.e., for the unstable manifold emanating from the separation point. This approximation is refined by the higher-order averaging of (17).

As we show in Appendix C, second-order averaging leads to an exact expression for the time-varying separation slope (the slope of the separation profile at the separation point). Specifically, for incompressible flows, at any time t_0, the tangent of the angle enclosed by the y axis and the separation profile at the boundary is given by

$$f_0(t_0)=\tan[\alpha(t_0)]=-\frac{\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\left[u_{yy}(\gamma,0,t)+3u_{xy}(\gamma,0,t)\int_{t_0}^{t}u_y(\gamma,0,s)ds\right]dt}{3\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}u_{xy}(\gamma,0,t)dt}, \qquad (20)$$

which agrees with the formula derived by Haller[4] for general incompressible flows. For the compressible version of (20), see formula (C9).[10]

V. AN EXAMPLE: UNSTEADY SEPARATION BUBBLE FLOW

In this section, we test our separation criterion on variants of an unsteady separation bubble model derived by Ghosh *et al.*,[7] who obtained the model from the Navier–Stokes equation using the perturbative procedure of Perry and Chong.[11]

A. Time-periodic incompressible separation bubble

We first consider the original velocity field derived by Ghosh *et al.*[7] for the study of passive scalar transport near an unsteady separation bubble. The velocity field is of the form

$$u(x,y,t)=-y+3y^2+x^2y-\tfrac{2}{3}y^3+\beta xy\sin\omega t,$$

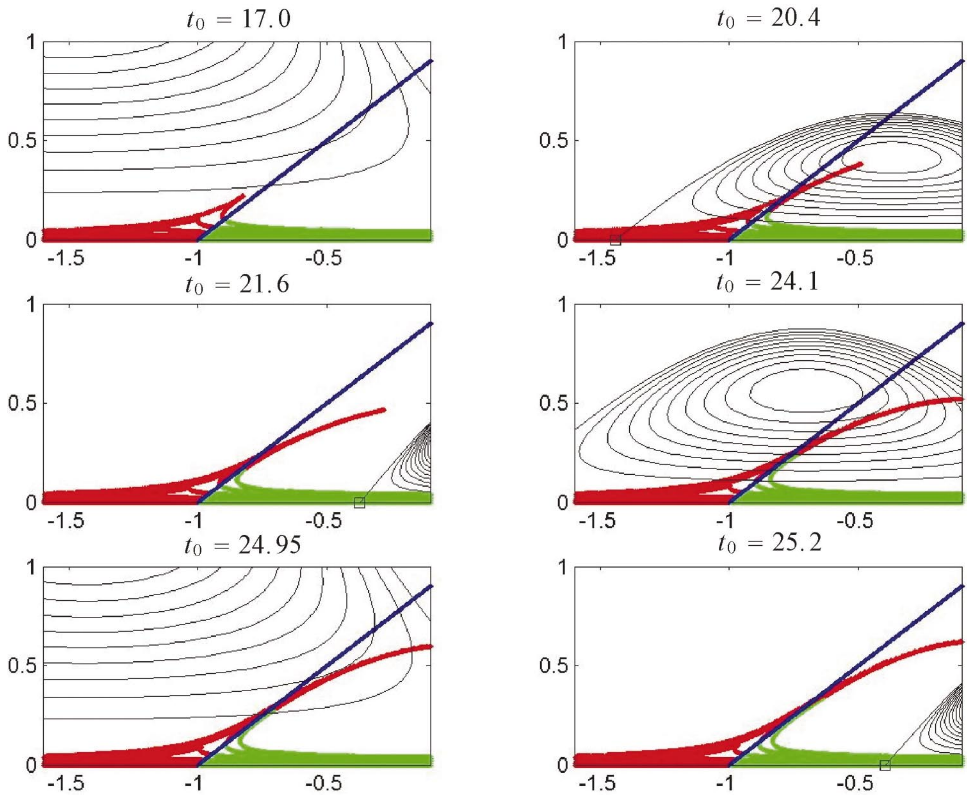

FIG. 3. (Color). Time-periodic separation bubble with parameters $\beta=3$ and $\omega=2\pi$. Particles with different colors are released from different sides of the predicted profile. We also plot instantaneous streamlines and the instantaneous zero of the skin friction (denoted by a small square).

$$v(x,y,t) = -xy^2 - \tfrac{1}{2}\beta y^2 \sin \omega t, \tag{21}$$

with the wall located at $y=0$. This velocity field is analytic and remains uniformly bounded in backward time on compact sets.

Verifying the assumptions of Sec. II A, we first note that

$$\mathbf{v}^0(\mathbf{x}) = \begin{pmatrix} -y + 3y^2 + x^2y - \dfrac{2}{3}y^3 \\ -xy^2 \end{pmatrix}, \tag{22}$$

thus the velocity field admits a finite asymptotic average on bounded sets of the (x,y) plane. Also, the function

$$\boldsymbol{\Delta}(\mathbf{x},t) = \frac{\beta(\cos \omega t_0 - \cos \omega t)}{\omega} \begin{pmatrix} xy \\ -\dfrac{1}{2}y^2 \end{pmatrix}$$

and its derivatives are uniformly bounded in time on bounded sets, thus our assumptions listed in Sec. II A are satisfied.

By the time-periodicity of (21), the incompressible version (8) of our sufficient separation criterion gives fixed unsteady separation at a point $(\gamma,0)$ if

$$\int_0^T u_y(\gamma,0,t)dt = 0, \quad \int_0^T u_{xy}(\gamma,0,t)dt < 0. \tag{23}$$

In this example, (23) yields the sufficient separation condition

$$-1 + \gamma^2 = 0, \quad 2\gamma < 0, \tag{24}$$

which in turn gives the fixed separation point location $\gamma=-1$. Thus, there exists a separation point at $(-1,0)$ in agreement with the numerical observations of Ghosh *et al.* By comparison, the sufficient separation criterion (4)–(6) of Haller[4] applied to this example gives

$$-1 + \gamma^2 = 0, \quad 2\gamma + \beta \sin \omega t < 0, \quad t \in \mathbb{R}, \tag{25}$$

and hence only guarantees separation at $\gamma=-1$ for $\beta<2$.

Calculating the separation slope from (20), we find that

$$f_0(t_0) = 1 + \frac{\beta}{\omega}\cos \omega t_0.$$

We show a numerical simulation in Fig. 3 for the case of $\beta=3$, in which conditions (25) fail, but our separation criterion correctly predicts fixed unsteady separation.

B. Quasiperiodic incompressible separation bubble

The velocity field,

$$u(x,y,t) = -y + 3y^2 + x^2y - \tfrac{2}{3}y^3 + xy[\beta_1 \sin \omega_1 t + \beta_2 \sin \omega_2 t],$$

$$v(x,y,t) = -xy^2 - \tfrac{1}{2}y^2[\beta_1 \sin \omega_1 t + \beta_2 \sin \omega_2 t], \tag{26}$$

is a quasiperiodic generalization of (21) when ω_1/ω_2 is irrational. Physically, (26) models the loss of stability of a steady separation bubble that develops oscillations with two dominant frequencies.

The averaged velocity $\mathbf{v}^0(\mathbf{x})$ is again equal to (22), and the function $\Delta(\mathbf{x},t)$ takes the form

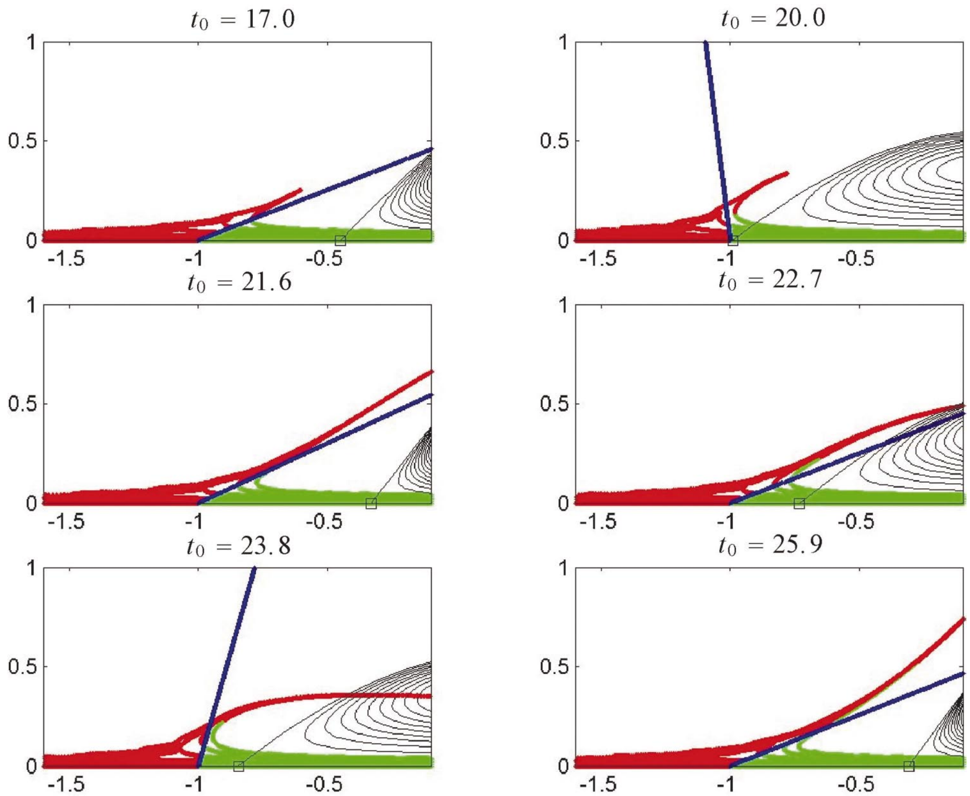

FIG. 4. (Color). Separation in the quasiperiodic incompressible separation bubble model. The parameters are $\beta_1=\beta_2=2$, $\omega_1=2\pi$, and $\omega_2=\sqrt{2}$.

$$\mathbf{\Delta}(\mathbf{x},t)=\left(\frac{\beta_1}{\omega_1}[\cos\,\omega_1 t_0-\cos\,\omega_1 t]+\frac{\beta_2}{\omega_2}[\cos\,\omega_2 t_0-\cos\,\omega_2 t]\right)\begin{pmatrix} xy \\ -\frac{1}{2}y^2 \end{pmatrix},$$

which is again uniformly bounded on bounded sets along with its spatial derivatives. Thus the hypotheses of Theorem 1 are again satisfied.

Applying the incompressible separation condition (8) to this example gives a result identical to (24), hence the flow separates at $(-1,0)$. This time, the separation slope formula (20) yields

$$f_0(t_0)=1+\frac{\beta_1}{\omega_1}\cos\,\omega_1 t_0+\frac{\beta_2}{\omega_2}\cos\,\omega_2 t_0.$$

We show a numerical verification of the separation location and slope in Fig. 4.

C. Quasiperiodic compressible separation bubble

We consider a compressible modification of (21) in the form

$$u(x,y,t)=-y+3y^2+x^2y-\tfrac{2}{3}y^3+\beta xy\sin\,\omega_1 t,$$

$$v(x,y,t)=-xy^2-\tfrac{1}{2}\beta y^2\sin\,\omega_1 t+\frac{\sin\,\omega_2 t}{\kappa-\cos\,\omega_2 t}y, \qquad (27)$$

with the parameter $\kappa>1$. For this strongly compressible velocity field, we have

$$\exp\left[\int_{t_0}^{t} v_y(x,0,s)ds\right]=\frac{(\kappa-\cos\,\omega_2 t)^{1/\omega_2}}{(\kappa-\cos\,\omega_2 t_0)^{1/\omega_2}}<\infty,$$

thus our assumption (10) on bounded densities is satisfied.

The averaged velocity field is again (22), and we have

$$\mathbf{\Delta}(\mathbf{x},t)=\begin{pmatrix} \beta/\omega(\cos\,\omega t_0-\cos\,\omega t)xy \\ -\beta/(2\omega)(\cos\,\omega t_0-\cos\,\omega t)y^2+y/\omega_2\,\mathrm{en}\dfrac{\kappa-\cos\,\omega_2 t}{\kappa-\cos\,\omega_2 t_0} \end{pmatrix},$$

thus $\mathbf{\Delta}(\mathbf{x},t)$ and its spatial derivatives are uniformly bounded on bounded sets. Thus all assumptions of Sec. II A are again satisfied.

Choosing $\omega_1=\pi$ and $\omega_2=1$ for concreteness, we obtain a quasiperiodic velocity field. The sufficient separation conditions (19) then predict fixed unsteady separation at a point $(\gamma,0)$ whenever

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\frac{\kappa-\cos t}{\kappa-\cos t_0}(-1+\gamma^2+\beta\gamma\sin\,\pi t)dt=\frac{\kappa(\gamma^2-1)}{\kappa-\cos t_0}=0,$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\frac{\kappa-\cos t}{\kappa-\cos t_0}(2\gamma+\beta\sin\,\pi t)dt=\frac{2\kappa\gamma}{\kappa-\cos t_0}<0,$$

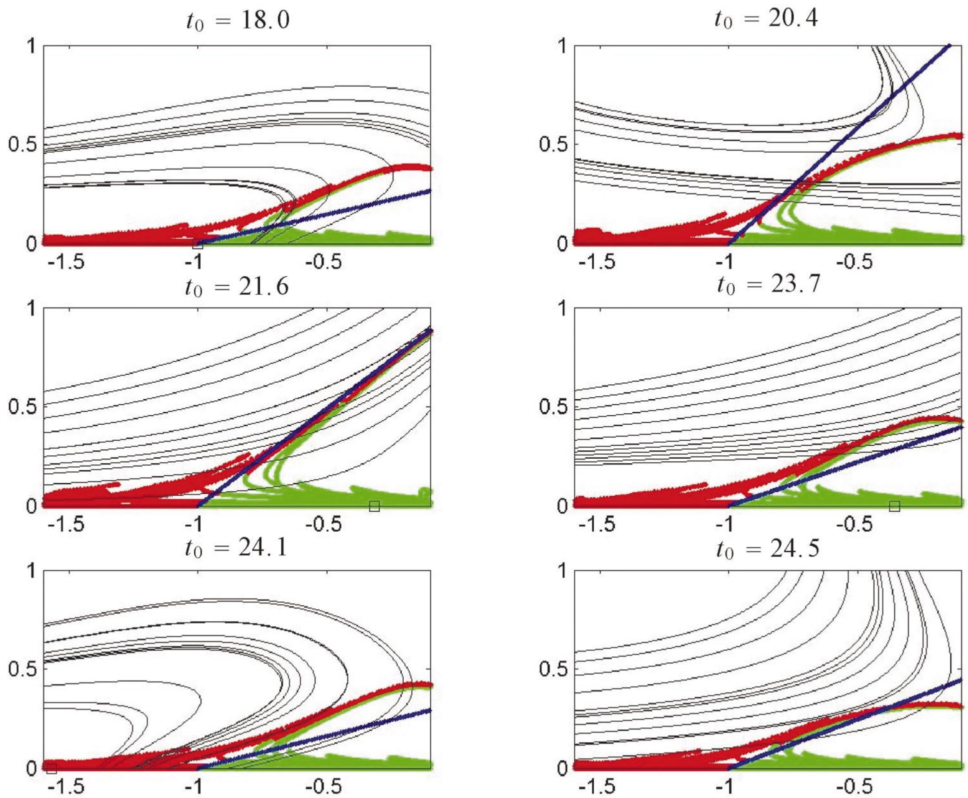

FIG. 5. (Color). Separation in the quasiperiodic compressible separation bubble model with $\beta=\kappa=3$.

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\frac{\kappa-\cos t}{\kappa-\cos t_0}(-2\gamma-\beta\sin\pi t)dt$$
$$=-\frac{2\kappa\gamma}{\kappa-\cos t_0}>0.$$

These conditions again imply that fixed unsteady separation takes place at $(-1,0)$.

Computing the general compressible slope formula (C9), we obtain

$$f_0(t_0)=\frac{\kappa+1/6}{\kappa-\cos t_0}+\frac{\beta}{\omega}\frac{\kappa\cos\pi t_0}{\kappa-\cos t_0}+\frac{\beta}{\kappa-\cos t_0}\left(\frac{\cos[(\omega+1)t_0/2]}{\omega+1}+\frac{\cos[(\omega-1)t_0/2]}{\omega-1}\right).$$

The simulations shown in Fig. 5 confirm this separation slope and the separation location.

D. Aperiodic incompressible separation bubble

Finally, we consider the aperiodic-in-time separation bubble flow

$$u(x,y,t)=-y+3y^2+x^2y-\tfrac{2}{3}y^3+\beta xy\sin(\pi t^2/2),$$

$$v(x,y,t)=-xy^2-\tfrac{1}{2}\beta y^2\sin(\pi t^2/2).$$

The Fresnel integral

$$S(t)=\int_0^t\sin\left(\frac{\pi}{2}\tau^2\right)d\tau$$

is approximated by

$$S(t)\sim\frac{1}{2}-\frac{1}{\pi t}\cos\left(\frac{\pi}{2}t^2\right)$$

for large values of t, which confirms the boundedness of (11) and (12) on bounded sets.

Following the same steps as in the previous examples, we find that the point $(-1,0)$ is a separation point and the separation slope is given by the constant $f_0=1$. We show the corresponding numerical simulations in Fig. 6.

E. Conclusions

In this paper, we have shown how the method of averaging can be used to strengthen the sufficient unsteady separation criterion of Haller[4] in the case of fluid flows with a well-defined asymptotic mean velocity. Such means certainly exist for the time-periodic and quasiperiodic flows, but also appear to be present in turbulent boundary layers.

The work presented here is one of the few known physical applications of first- and second-order aperiodic averaging. We stress that we applied averaging without an adiabatic assumption on the velocity field. The slowly varying nature of the separation problem arises from the presence of the no-slip boundary.

The application of averaging methods to moving unsteady separation remains an open question. In moving separation, the location of the separation point changes in time,

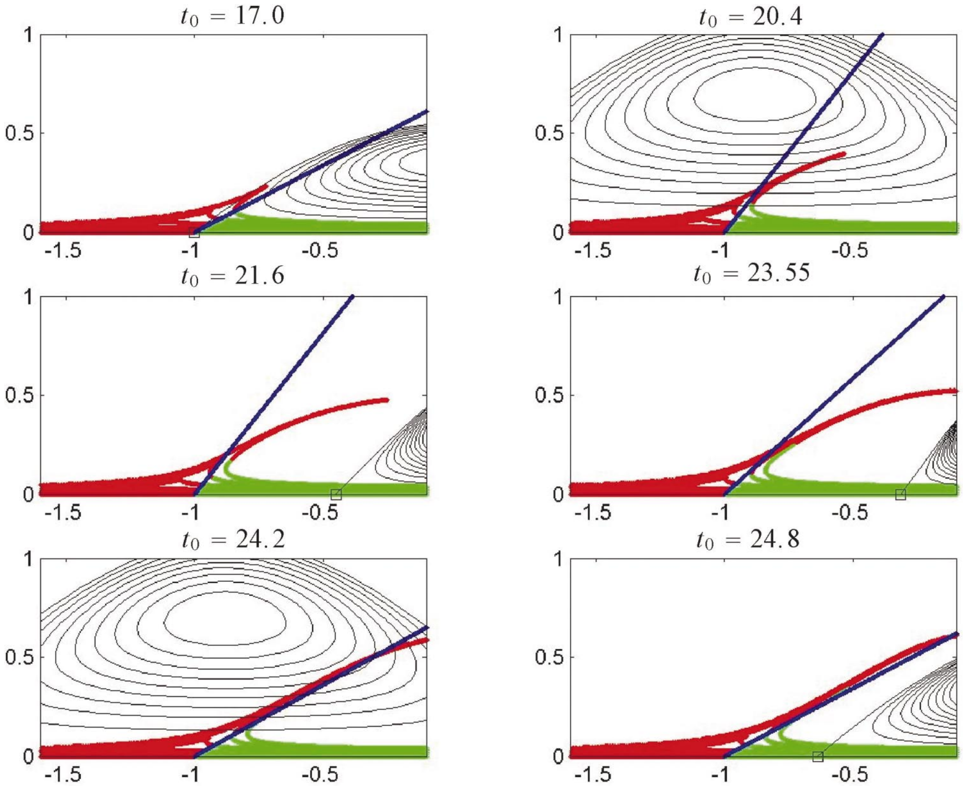

FIG. 6. (Color). Fixed unsteady separation in the aperiodic separation bubble model.

and hence will not be captured by averaging over infinite times. Still, an appropriately modified finite-time averaging method may lead to an improvement of the moving separation criterion of Haller.[4]

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APPENDIX A

In this and later appendixes, we assume that the velocity field $\mathbf{v}(\mathbf{x},t)$ is at least three-times continuously differentiable in its arguments. This assumption is needed for our forthcoming changes of variables to be well defined.

To derive the normal form (17), we first introduce the near-identity change of variables

$$\mathbf{x} = \boldsymbol{\xi} + \epsilon \mathbf{w}(\boldsymbol{\xi},t), \tag{A1}$$

with $\boldsymbol{\xi}=(\xi,\eta)$, and with a uniformly bounded function $\mathbf{w}$ to be specified later. We then rewrite (15) as

$$\dot{\mathbf{x}} = \dot{\boldsymbol{\xi}} + \epsilon D_{\boldsymbol{\xi}}\mathbf{w}\dot{\boldsymbol{\xi}} + \epsilon\frac{\partial \mathbf{w}}{\partial t} = \epsilon \mathbf{f}(\boldsymbol{\xi}+\epsilon\mathbf{w},t) + \epsilon^2 \mathbf{g}(\boldsymbol{\xi}+\epsilon\mathbf{w},t)$$

$$= \epsilon\mathbf{f}(\boldsymbol{\xi},t) + \epsilon^2[D_{\boldsymbol{\xi}}\mathbf{f}(\boldsymbol{\xi},t)\mathbf{w} + \mathbf{g}(\boldsymbol{\xi},t)] + \mathcal{O}(\epsilon^3), \tag{A2}$$

where the $\mathcal{O}(\epsilon^3)$ term remains bounded by the assumptions of Sec. II A. for all $t \leq t_0$, with t_0 arbitrary but fixed.

From the transformed equations (A2), we obtain

$$(\mathbf{I} + \epsilon D_{\boldsymbol{\xi}}\mathbf{w})\dot{\boldsymbol{\xi}} = \epsilon\left[\mathbf{f}(\boldsymbol{\xi},t) - \frac{\partial \mathbf{w}}{\partial t}\right] + \epsilon^2[D_{\boldsymbol{\xi}}\mathbf{f}(\boldsymbol{\xi},t)\mathbf{w} + \mathbf{g}(\boldsymbol{\xi},t)] + \mathcal{O}(\epsilon^3).$$

If $\|D_{\boldsymbol{\xi}}\mathbf{w}\|$ remains uniformly bounded for all $t \leq t_0$, then, for small enough ϵ, we obtain

$$\dot{\boldsymbol{\xi}} = \epsilon(\mathbf{I} + \epsilon D_{\boldsymbol{\xi}}\mathbf{w})^{-1}\left[\mathbf{f}(\boldsymbol{\xi},t) - \frac{\partial \mathbf{w}}{\partial t} + \epsilon D_{\boldsymbol{\xi}}\mathbf{f}(\boldsymbol{\xi},t)\mathbf{w} + \epsilon\mathbf{g}(\boldsymbol{\xi},t)\right] + \mathcal{O}(\epsilon^3).$$

Using

$$(\mathbf{I} + \epsilon D_{\boldsymbol{\xi}}\mathbf{w})^{-1} = I - \epsilon D_{\boldsymbol{\xi}}\mathbf{w} + \mathcal{O}(\epsilon^2),$$

we obtain the equivalent system

$$\dot{\boldsymbol{\xi}} = \epsilon\left[\mathbf{f}(\boldsymbol{\xi},t) - \frac{\partial \mathbf{w}}{\partial t}\right] + \epsilon^2\left[D_{\boldsymbol{\xi}}\mathbf{f}(\boldsymbol{\xi},t)\mathbf{w} - D_{\boldsymbol{\xi}}\mathbf{w}\ \mathbf{f}(\xi,t) + D_{\boldsymbol{\xi}}\mathbf{w}\frac{\partial \mathbf{w}}{\partial t} + \mathbf{g}(\boldsymbol{\xi},t)\right] + \mathcal{O}(\epsilon^3). \tag{A3}$$

In general, we cannot eliminate $\mathcal{O}(\epsilon)$ for any choice of $\mathbf{w}$, because $\mathbf{f}(\boldsymbol{\xi},t)$ has a (generally nonzero) asymptotic mean by assumption, whereas $\partial \mathbf{w}/\partial t$ always has zero asymptotic mean. Instead, we select $\mathbf{w}$ to satisfy

$$\mathbf{w}(\boldsymbol{\xi},t) = \int_{t_0}^{t} [\mathbf{f}(\boldsymbol{\xi},\tau) - \mathbf{f}^0(\boldsymbol{\xi})] d\tau, \quad \mathbf{f}^0(\boldsymbol{\xi}) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0-T}^{t_0} \mathbf{f}(\boldsymbol{\xi}) dt, \tag{A4}$$

which turns (A3) into

$$\dot{\boldsymbol{\xi}} = \epsilon \mathbf{f}^0(\boldsymbol{\xi}) + \epsilon^2 [D_{\boldsymbol{\xi}}\mathbf{f}(\boldsymbol{\xi},t)\mathbf{w} - D_{\boldsymbol{\xi}}\mathbf{w}\, \mathbf{f}^0(\boldsymbol{\xi},t) + \mathbf{g}(\boldsymbol{\xi},t)] + \mathcal{O}(\epsilon^3), \tag{A5}$$

as claimed in (17).

It remains to note that the $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon^3)$ terms in (A5) remain uniformly bounded in time if $\mathbf{w}$, as well as its derivatives up to second order, remain uniformly bounded for $t \leqslant t_0$. By (A4), $\|D^k_{\boldsymbol{\xi}}\mathbf{w}\|$ remains uniformly bounded for $k=0$, 1, 2, 3, if

$$\limsup_{t\to-\infty} \left\| \int_{t_0}^{t} D^k_{\boldsymbol{\xi}} [\mathbf{f}(\boldsymbol{\xi},\tau) - \mathbf{f}^0(\boldsymbol{\xi})] d\tau \right\| < \infty$$

remains uniformly bounded for the same k. But this last condition is satisfied by the last assumption of Sec. II A.

APPENDIX B

To prove Theorem 1, we first rewrite the averaged normal form (17) as

$$\dot{\xi} = \epsilon\eta[a^0(\xi) + \mathcal{O}(\epsilon\eta)],$$

$$\dot{\eta} = \epsilon\eta^2[c^0(\xi) + \mathcal{O}(\epsilon\eta)], \tag{B1}$$

with

$$a^0(\xi) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0-T}^{t_0} A_2(\xi,0,t) dt,$$

$$c^0(\xi) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0-T}^{t_0} C_2(\xi,0,t) dt. \tag{B2}$$

Recall that by assumption (19), we have

$$a^0(\gamma) = 0, \quad a^0_{\xi}(\gamma) < 0, \quad c^0(\gamma) > 0. \tag{B3}$$

Next, we localize (B1) in the ξ direction near the candidate separation point $\mathbf{P}=(\gamma,0)$ by introducing the new variable

$$\zeta = \xi - \gamma.$$

Rewriting (B1) in terms of the (ζ,η) variables, we obtain

$$\dot{\zeta} = \epsilon\eta\zeta[a^0_{\xi}(\gamma) + \mathcal{O}(\epsilon\eta) + \mathcal{O}(\zeta)],$$

$$\dot{\eta} = \epsilon\eta^2[c^0(\gamma) + \mathcal{O}(\epsilon\eta) + \mathcal{O}(\zeta)], \tag{B4}$$

where the terms $\mathcal{O}(\epsilon\eta)$ and $\mathcal{O}(\zeta)$ are uniformly bounded in the vicinity of $(\zeta,\eta)=(0,0)$ for all $t\leqslant t_0$, by the assumptions of Sec. II A.

Following Ref. 4, we shall construct an unstable manifold (the separation profile) for system (B4) in the cone

$$Q = \{(\zeta,\eta) : |\zeta| \leqslant \eta,\ 0 \leqslant \eta \leqslant \beta\}, \tag{B5}$$

where β is a positive constant to be selected below.

Along the $\eta=\beta$ boundary of Q, we obtain from Eq. (B4) and assumption (B3) that

$$\dot{\eta}\big|_{\eta=\beta} = \epsilon\beta^2[c^0(\gamma) + \mathcal{O}(\epsilon) + \mathcal{O}(\zeta)] \geqslant \tfrac{1}{2}\epsilon\beta^2 c^0(\gamma) > 0, \tag{B6}$$

provided that we select ϵ and β small enough. For such choices of ϵ and β, (B6) shows that trajectories intersecting the $\eta=\beta$ boundary of Q leave Q immediately.

Next, we consider the $\zeta=\eta$ boundary of the cone Q, on which, by (B3) and (B4), we have the estimate

$$\dot{\zeta}\big|_{\zeta=\eta} = \epsilon\zeta\eta[a^0_{\xi}(\gamma) + \mathcal{O}(\epsilon\eta) + \mathcal{O}(\zeta)] < \tfrac{1}{2}\epsilon\zeta\eta a^0_{\xi}(\gamma) < 0 \tag{B7}$$

for all $t\leqslant t_0$, provided that ϵ and β are small enough (implying that $\eta\in[0,\beta]$ is small).

Equation (B4) and assumption (B3) also reveal that for $\epsilon,\beta>0$ sufficiently small,

$$\dot{\eta}\big|_{\zeta=\eta} = \epsilon\eta^2[c^0(\gamma) + \mathcal{O}(\epsilon) + \mathcal{O}(\zeta)] > \tfrac{1}{2}\epsilon\eta^2 c^0(\gamma) > 0, \tag{B8}$$

thus trajectories intersecting the $\zeta=\eta$ boundary of the cone Q immediately enter the cone. An identical argument establishes the same conclusion for the $\zeta=-\eta$ boundary of Q.

Based on (B6)–(B8), we conclude that for ϵ and β sufficiently small, the extended form of (B4),

$$\dot{\zeta} = \epsilon\zeta\eta a^0_{\xi}(\gamma) + \epsilon^2\eta^2 m_1(\zeta,\eta,t,\epsilon) + \epsilon\eta\zeta^2 m_2(\zeta,\eta,t,\epsilon),$$

$$\dot{\eta} = \epsilon\eta^2 c^0(\gamma) + \epsilon\eta m_3(\zeta,\eta,t,\epsilon) + \zeta m_4(\zeta,\eta,t,\epsilon),$$

$$\dot{t} = 1, \tag{B9}$$

has trajectories with the following properties on the closed set $\mathcal{Q}=Q\times\mathbb{R}$.

(a) The set of initial particle positions (ζ_0,η_0,t_0) that immediately leave $\mathcal{Q}$ in *backward time* is given by

$$W^{\mathrm{im}} = \{(\zeta,\eta,t) \in \mathcal{Q}: |\zeta| = \eta,\ \eta \in (0,\beta]\},$$

which is the union of the two disjoint components

$$W^{\mathrm{im}}_{\pm} = \{(\zeta,\eta,t) \in \mathcal{Q}: \zeta = \pm\eta,\ \eta \in (0,\beta]\}.$$

(b) If W^{ev} denotes the set of initial conditions (ζ_0,η_0,t_0) that eventually leave $\mathcal{Q}$ in backward time, then W^{im} is a relatively closed subset of W^{ev}. In other words, if a sequence within $W^{\mathrm{im}}_{\pm}$ converges to a point outside $W^{\mathrm{im}}_{\pm}$, then that point is necessarily at $\zeta=\eta=0$, which is outside W^{ev}.

The properties (a) and (b), by definition, make $\mathcal{Q}$ a backward-time *Wazewski set* for the extended system (B9) (cf. Hale[12]). As a result, the Wazewski principle holds for $\mathcal{Q}$: the map $\Gamma: W^{\mathrm{ev}} \to W^{\mathrm{im}}$ that maps initial positions in $\mathcal{Q}$ to their point of exit in backward time is continuous.

Using the Wazewski principle, we want to argue that there are trajectories that never leave $\mathcal{Q}$ in backward time once they enter it. Assume the contrary, i.e., assume now that all initial conditions in $\mathcal{Q}$ eventually leave $\mathcal{Q}$ in backward time. This would imply

$$W^{\mathrm{ev}} = \mathcal{Q} - \{\zeta=\eta=0\},$$

which would be a contradiction, because then W^{ev} would be connected, and hence could not be mapped by a continuous map Γ into the disconnected set W^{im}.

We therefore conclude that $W^{\mathrm{ev}} \neq \mathcal{Q}-\{\zeta=\eta=0\}$, i.e., there is a nonempty set W^{∞} of initial fluid particle positions that stay in $\mathcal{Q}$ for all backward times. By definition, W^{∞} is an invariant set and is necessarily smooth in t, because it is composed of fluid trajectories that are smooth in t.

Next we want to argue that all solutions in W^{∞} tend to $\zeta=\eta=0$ in backward time. Consider a specific initial condition $(\zeta_0,\eta_0,t_0) \in W^{\infty}$, and denote the trajectory emanating from this initial position by $[\zeta(t),\eta(t),t]$. Along this trajectory, we have

$$\dot{\eta}(t) = \epsilon\eta^2[c^0(\gamma) + \epsilon\eta m_3(\zeta,\eta,t,\epsilon) + \zeta m_4(\zeta,\eta,t,\epsilon)],$$

which, upon integration, gives

$$\eta(t) = \frac{\eta_0}{1+\epsilon\int_t^{t_0}[c^0(\gamma)+\epsilon\eta m_3(\zeta,\eta,s,\epsilon)+\zeta m_4(\zeta,\eta,s,\epsilon)]ds} \leqslant \frac{\eta_0}{1+\epsilon\int_t^{t_0}[c^0(\gamma)-\epsilon\beta|m_3(\zeta,\eta,s,\epsilon)|-\beta|m_4(\zeta,\eta,s,\epsilon)|]ds}.$$

This last equation holds for all $t \leqslant t_0$, because the trajectory we consider stays in $\mathcal{Q}$ for all backward times. For small enough ϵ and β, the uniform boundedness of $m_k(\zeta,\eta,s,\epsilon)$ within $\mathcal{Q}$ leads to the estimate

$$\eta(t) \leqslant \frac{\beta}{1+\epsilon\int_t^{t_0}\frac{1}{2}c^0(\gamma)d\tau} = \frac{\beta}{1+\frac{1}{2}\epsilon c^0(\gamma)(t_0-t)}, \qquad \text{(B10)}$$

allowing us to conclude that

$$\lim_{t\to-\infty}\eta(t) = 0.$$

In other words, trajectories that never leave $\mathcal{Q}$ in backward time will necessarily converge to the $\eta=0$ boundary of the cone $\mathcal{Q}$. By the definition of $\mathcal{Q}$, however, this convergence in the η direction implies

$$\lim_{t\to-\infty}\zeta(t) = 0.$$

We therefore conclude that all trajectories in W^{∞} converge to $\eta=\zeta=0$ in backward time, thus W^{∞} is an unstable manifold for $\mathbf{p}=(\gamma,0)$ for all $t \leqslant t_0$.

APPENDIX C

Here we derive the incompressible separation slope formula (20) by second-order averaging of the normal form (17). For the following, we need to assume that the velocity field $\mathbf{v}(\mathbf{x},t)$ in (2) is a class C^r function with $r \geqslant 4$. First, supplementing the notation introduced in (B2) with

$$a(\xi,t) = A_2(\xi,0,t),$$

$$c(\xi,t) = C_2(\xi,0,t),$$

$$\phi(\xi,t) = \int_{t_0}^{t}[a(\xi,s)-a^0(\xi)]ds,$$

$$\psi(\xi,t) = \int_{t_0}^{t}[c(\xi,s)-c^0(\xi)]ds, \qquad \text{(C1)}$$

we rewrite the averaged normal form (17) as

$$\dot{\boldsymbol{\xi}} = \epsilon \mathbf{f}^0(\boldsymbol{\xi}) + \epsilon^2 \mathbf{f}^1(\boldsymbol{\xi},t) + \mathcal{O}(\epsilon^3), \tag{C2}$$

with

$$\mathbf{f}^0(\boldsymbol{\xi}) = [\eta a^0(\xi), \eta^2 c^0(\xi)],$$

$$\mathbf{f}^1(\boldsymbol{\xi},t) = D_{\boldsymbol{\xi}}\mathbf{f}(\boldsymbol{\xi},t)\mathbf{w} - D_{\boldsymbol{\xi}}\mathbf{w}\mathbf{f}^0(\boldsymbol{\xi}) + \mathbf{g}(\boldsymbol{\xi},t;0),$$

$$\mathbf{f}(\boldsymbol{\xi},t) = \begin{pmatrix} \eta a(\xi,t) \\ \eta^2 c(\xi,t) \end{pmatrix},$$

$$\mathbf{w}(\boldsymbol{\xi},t) = \begin{pmatrix} \eta \phi(\xi,t) \\ \eta^2 \psi(\xi,t) \end{pmatrix},$$

$$\mathbf{g}(\boldsymbol{\xi},t;0) = \begin{pmatrix} \eta^2 D_y A_2(\xi,0,t) \\ \eta^3 D_y C_2(\xi,0,t) \end{pmatrix}.$$

Evaluating the function $\mathbf{f}^1(\boldsymbol{\xi},t)$, we find that

$$\mathbf{f}^1 = \begin{pmatrix} \eta a_\xi & a \\ \eta^2 c_\xi & 2\eta c \end{pmatrix} \begin{pmatrix} \eta\phi \\ \eta^2\psi \end{pmatrix} - \begin{pmatrix} \eta\phi_\xi & \phi \\ \eta^2\psi_\xi & 2\eta\psi \end{pmatrix} \begin{pmatrix} \eta a^0 \\ \eta^2 c^0 \end{pmatrix} + \begin{pmatrix} \eta^2 D_y A_2(\xi,0,t) \\ \eta^3 D_y C_2(\xi,0,t) \end{pmatrix} = \begin{pmatrix} \eta^2 F \\ \eta^3 G \end{pmatrix},$$

with

$$F(\xi,t) = (a_\xi - c^0)\phi + a\psi - \phi_\xi a^0 + D_y A_2(\xi,0,t),$$

$$G(\xi,t) = c_\xi \phi + 2(c - c^0)\psi - \psi_\xi a^0 + D_y C_2(\xi,0,t).$$

With the above form of $\mathbf{f}^1(\boldsymbol{\xi},t)$ at hand, we perform second-order averaging on system (C2) by seeking a near-identity coordinate change

$$\boldsymbol{\xi} = \boldsymbol{\mu} + \epsilon^2 \mathbf{h}(\boldsymbol{\mu},t), \quad \boldsymbol{\mu} = (\mu,\lambda) \tag{C3}$$

that removes the explicit time dependence in (C2) at order $\mathcal{O}(\epsilon^2)$. As in the case of first-order averaging, we eliminate the oscillatory part of $\mathbf{f}^1(\boldsymbol{\xi},t)$ by picking $\mathbf{h}$ appropriately. We then obtain the second-order averaged normal form

$$\dot{\mu} = \epsilon\lambda[a^0(\mu) + \epsilon\lambda F^0(\mu)] + \mathcal{O}(\epsilon^3\lambda^3),$$

$$\dot{\lambda} = \epsilon\lambda^2[c^0(\mu) + \epsilon\lambda G^0(\mu)] + \mathcal{O}(\epsilon^3\lambda^4), \tag{C4}$$

where

$$F^0(\xi) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0-T}^{t_0} [(a_\xi - c^0)\phi + a\psi - \phi_\xi a^0 + D_y A_2(\xi,0,t)]dt,$$

$$G^0(\xi) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0-T}^{t_0} [c_\xi\phi + 2(c - c^0)\psi - \psi_\xi a^0 + D_y C_2(\xi,0,t)]dt. \tag{C5}$$

Neglecting the time-dependent terms in the normal form (17) and rescaling time by letting $d\tau/dt = \epsilon\lambda(t)$, we obtain the system

$$\mu' = a^0(\mu) + \epsilon\lambda F^0(\mu),$$

$$\lambda' = \lambda c^0(\mu) + \epsilon\lambda^2 G^0(\mu), \tag{C6}$$

which, under the conditions of Theorem 1, has an unstable manifold at $(\mu,\lambda)=(\gamma,0)$. This hyperbolic unstable manifold is tangent to the unstable eigenvector

$$\mathbf{e} = \begin{pmatrix} \epsilon F^0(\gamma) \\ c^0(\gamma) - a_\xi^0(\gamma) \end{pmatrix}.$$

Recalling that at time $t=t_0$,

$$(\mu,\lambda) = (\xi,\eta) + \mathcal{O}(\epsilon^2) = (x,\tilde{y}/\epsilon) + \mathcal{O}(\epsilon) = (x,y/\epsilon) + \mathcal{O}(\epsilon),$$

we conclude that the slope of the eigenvector $\mathbf{e}$ relative to the y axis in the original (x,y) coordinates is given by

$$f_0(t_0) = \frac{F^0(\gamma)}{c^0(\gamma) - a_\xi^0(\gamma)}, \tag{C7}$$

where all the three averaged quantities depend on t_0, the starting point of the asymptotic averaging operation

$$(\cdot)^0 = \lim_{t\to\infty} \frac{1}{T}\int_{t_0-T}^{t_0} (\cdot)dt.$$

To evaluate (C7), we first note that by (C1),

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0-T}^{t_0} a(\gamma,t)\psi(\gamma,t)dt = \lim_{T\to\infty} \frac{1}{T}\left\{ [\phi(\gamma,t)\psi(\gamma,t)]_{t_0-T}^{t_0} - \int_{t_0-T}^{t_0} \phi(\gamma,t)[c(\gamma,t) - c^0(\gamma)dt] \right\}$$

$$= -\lim_{T\to\infty} \frac{1}{T}\int_{t_0-T}^{t_0} \phi(\gamma,t)[c(\gamma,t) - c^0(\gamma)]dt, \tag{C8}$$

where we used $a^0(\gamma)=0$ and the uniform boundedness of ϕ and ψ in time. Using (C5) and (C8), we obtain

$$
\begin{aligned}
F^0(\gamma) &= \lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}[(a_\xi - c^0)\phi - \phi_\xi a^0 + a\psi + D_y A_2(\xi,0,t)]_{\xi=\gamma}dt \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\left[(a_\xi - c^0)\int_{t_0}^{t} a\ ds - (c-c^0)\int_{t_0}^{t} a\ ds + D_y A_2(\xi,0,t)\right]_{\xi=\gamma} dt \\
&= \lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\left[(a_\xi - c)\int_{t_0}^{t} a\ ds + D_y A_2(\xi,0,t)\right]_{\xi=\gamma} dt,
\end{aligned}
$$

which together with (C7) yields the slope formula

$$
f_0(t_0) = \frac{\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\left\{D_y A_2(\gamma,0,t) + [a_\xi(\gamma,t) - c(\gamma,t)]\int_{t_0}^{t} a(\gamma,s)ds\right\}dt}{\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}[c(\gamma,t) - a_\xi(\gamma,t)]d\tau} \tag{C9}
$$

in the $(x,\tilde{y})$ coordinates.

For incompressible flows, $y=\tilde{y}$ for all t, because $v_y(\gamma,0,t)=0$ in (14). We then obtain from (C9) the separation slope

$$
\begin{aligned}
f_0(t_0) &= \frac{\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\left\{\frac{1}{2}u_{yy}(\gamma,0,t) + \left[u_{xy}(\gamma,0,t) - \frac{1}{2}v_{yy}(\gamma,0,t)\right]\int_{t_0}^{t} u_y(\gamma,0,s)ds\right\}dt}{\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\left[\frac{1}{2}v_{yy}(\gamma,0,t) - u_{xy}(\gamma,0,t)\right]dt} \\
&= -\frac{\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\left[u_{yy}(\gamma,0,t) + 3u_{xy}(\gamma,0,t)\int_{t_0}^{t} u_y(\gamma,0,s)ds\right]dt}{3\lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0} u_{xy}(\gamma,0,t)dt}
\end{aligned}
$$

in the original (x,y) coordinate system, as claimed in (20).

We finally note that $\mathbf{h}(\boldsymbol{\mu},t)$ must be uniformly bounded for all backward times for the second-order averaged system to be well defined. Because $\mathbf{h}(\boldsymbol{\mu},t)$ is selected so that

$$
\begin{aligned}
\frac{\partial \mathbf{h}}{\partial t} &= \begin{pmatrix} F(x,t) - F^0(x) \\ G(x,t) - G^0(x) \end{pmatrix} \\
&= \begin{pmatrix} a_\xi\phi + a\psi - \phi c^0 + D_y A_2(\xi,0,t) \\ c_\xi\phi + 2(c-c^0)\psi + D_y C_2(\xi,0,t) \end{pmatrix} \\
&\quad - \lim_{T\to\infty}\frac{1}{T}\int_{t_0-T}^{t_0}\begin{pmatrix} a_\xi\phi + a\psi - \phi c^0 + D_y A_2(\xi,0,t) \\ c_\xi\phi + 2(c-c^0)\psi + D_y C_2(\xi,0,t) \end{pmatrix}dt,
\end{aligned}
$$

we require that the integrals

$$
\begin{aligned}
&\int_{t_0}^{t}[(a_\xi - c^0)\phi + D_y A_2(\xi,0,s)] - \{[(a_\xi - c^0)\phi]^0 + D_y A_2^0(\xi,0)\}ds, \\
&\int_{t_0}^{t}[c_\xi\phi + 2(c-c^0)\psi + D_y C_2(\gamma,0,s) - \{(c_\xi\phi)^0 + 2[(c-c^0)\psi]^0 + D_y C_2^0(\xi,0)\}]ds,
\end{aligned} \tag{C10}
$$

remain uniformly bounded as $t\to-\infty$, in addition to our previous assumptions.

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