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Multi-Dimensional Homoclinic Jumping and the Discretized NLS Equation

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Abstract: We consider a class of dynamical systems that arise frequently in multi-mode truncations and discretizations of partial differential equations, including the perturbed NLS. We develop a general method to detect the existence of multi-pulse solutions that are doubly asymptotic to an invariant manifold with two different time scales. We use our method together with some recent results of Li and McLaughlin to show the existence of several families of multi-pulse orbits for the Ablowitz-Ladik discretization of the perturbed NLS. These orbits include N-pulse heteroclinic orbits and N-pulse Šilnikov-type orbits for arbitrarily large N.

1. Introduction

In this paper we study a class of multi-degree-of-freedom dynamical systems which arise in modal truncations of partial differential equations on periodic domains. One usually arrives at these equations when looking for small amplitude solutions of a PDE with parametric forcing terms. An important prototype example is the damped-forced sine-Gordon equation, which we discuss briefly below for motivation.

As shown in, e.g., Bishop *et al.* [5], a small amplitude approximation to the sine-Gordon equation leads to a perturbed nonlinear Schrödinger Eq. (NLS). For a range of parameters, the integrable limit of the NLS admits one linearly stable and one unstable mode together with infinitely many neutrally stable modes. These latter modes can be further decomposed into a mode of plane waves (i.e., solutions with no spatial structure) and an infinite number of neutrally stable, i.e., oscillatory modes. A finite dimensional approximation to the problem is a well-known discretization of the NLS that produces an *integrable* system in the unperturbed limit (see Ablowitz and Ladik [1], Bogolyubov and Prikarpatskii [7], and Miller *et al.* [36]).

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In the discretized NLS the plane of spatially independent solutions is invariant under both the perturbed and the unperturbed dynamics. For zero dissipation and forcing, the plane contains a circle of fixed points which is surrounded by a one-parameter family of periodic solutions. Furthermore, the invariant plane lies in a codimension two center manifold that accounts for the non-planar oscillatory modes. The center manifold is normally hyperbolic as it admits a one-dimensional stable and a one-dimensional unstable subspace at each of its points. This hyperbolicity is due to the presence of the stable and unstable modes mentioned above, and gives rise to codimension one stable and unstable manifolds to the center manifold. These invariant manifolds then coincide in two homoclinic manifolds in the integrable limit of zero forcing and damping. This phase space geometry is quite remarkable as it is a precise finite dimensional model of the phase space structure of the original PDE (see, e.g., Ercolani *et al.* [8], Ercolani and McLaughlin [9], and Li and McLaughlin [30] for details).

A similar analogy exists between the phase space structure of the perturbed NLS equation and its two-mode approximation (see Bishop *et al.* [5, 6]). This fact inspired a great deal of work on modal truncations of the perturbed NLS, although all rigorous results so far are only concerned with the two-mode approximation that excludes the oscillatory modes (see Bishop *et al.* [5, 6], Kovačič and Wiggins [27], Haller and Wiggins [16], McLaughlin *et al.* [34], and Haller and Wiggins [19]). Other examples with the modal truncations of the same class include parametrically forced surface wave problems (Holmes [22], and Kambe and Umeki [25]), the dynamics of forced and damped thin plates (Feng and Sethna [10]), inextensional beams (Nayfeh and Pai [37], Feng and Leal [12]), and resonantly driven coupled pendula (Miles [35], Becker and Miles [4], and Kovačič and Wettergren [28]). All these problems can be recast in the form of Eq. (1) below. Our basic goal in this paper is to study the existence of nontrivial homoclinic and heteroclinic behavior in these systems by including an arbitrary high but finite number of modes.

The main result of the paper is the construction of a class of complicated solutions in multi-mode truncations or discretizations. These solutions admit three different time scales and correspond to irregular "jumping" around the plane Π of spatially independent modes. In our general formulation we in fact allow for the presence of a $2m$-dimensional manifold Π which contains an m-torus of equilibria in the unperturbed limit.

In backward time the solutions we construct asymptote to some set in Π which is born out of the perturbation of the torus of fixed points of the unperturbed limit. In forward time, after making several jumps away from Π, the solutions asymptote to other structures in the center manifold that lie in the vicinity of the manifold Π. We give a criterion for the existence of such solutions, which is a generalization of the *energy-phase method* developed in Haller [15] and Haller and Wiggins [19] for two-degree-of-freedom systems.

Under certain conditions, the solutions we construct will ultimately asymptote to some invariant set within the manifold Π. If their ω and α-limit sets coincide, then we obtain a multi-pulse orbit homoclinic to this set. An important special case arises when this set is an equilibrium that is a sink for the dynamics on the codimension two center manifold. We call the resulting multi-pulse orbit an *N-pulse Šilnikov-type orbit*. Such orbits seem to have a prominent role in creating complicated or chaotic dynamics in modal equations. While single-pulse Šilnikov orbits can also be obtained in these problems applying a modified Melnikov method (see Kovačič and Wiggins [27], Feng and Sethna [10], Feng and Wiggins [11], Tien and Namachchivaya [38], Kovačič and Wettergren [28]), and Li and McLaughlin [31], such orbits generically exist for a single codimension one surface in the space of system parameters. In contrast, our methods typically yield multi-pulse Šilnikov-type orbits on an intricate web of the parameter space (see Haller and Wiggins [19] for a two-mode example).

The main techniques we use in this paper include the perturbation theory of normally hyperbolic invariant manifolds, their stable and unstable manifolds, and stable and unstable foliations. We do not explicitly assume that in the limit of zero forcing and damping the modal equations are integrable. We do, however, assume the presence of particular structures in this limiting geometry, which are not typical in nonintegrable cases. Our strategy is to follow trajectories in the unstable manifold of the manifold Π as they leave and repeatedly return to a neighborhood of the center manifold. The control over individual trajectories is achieved by obtaining estimates on their location as well as on their energies before and after their intermediate passages near the center manifold. This amounts to studying the properties of an appropriately defined local Poincaré map. The results of this study are summarized in the Passage Lemma (Lemma 7.1), which sets the stage for a final implicit function argument in Theorem 7.3 of Sect. 6. This argument is subtle since the equation satisfied by multi-pulse homoclinic orbits becomes undefined in the limit of the vanishing perturbation parameter. We circumvent this problem by defining an extension to the local map at this limit, and use the Passage Lemma to conclude that this extension is of class C^1. We use the main result formulated in Theorem 7.3 on multi-pulse orbits to give conditions for the existence of multi-pulse orbits homoclinic to the manifold Π in Theorems 7.4-8.1 of Sect. 6. We study the "disintegration" of the unstable manifold of the plane Π via repeated jumping in Sect. 7. We give a useful reformulation of our method in Sect. 8 for the case when one of the invariants of the unperturbed limit is more convenient to use than the unperturbed Hamiltonian. An application of the results to a near-integrable discretization of the perturbed NLS is given in Sect. 9. Finally, we present some conclusions in Sect. 10.

2. Setting and assumptions

The class of modal truncations listed in the Introduction can be written in the general form

$$\dot{x} = \omega^\sharp[DH_0(x) + \epsilon DH_1(x)] + \epsilon g(x), \tag{1}$$

where $x \in \mathcal{P} \subset \mathbb{R}^{2(n+m+1)}$, with $n \geq 0$, $m \geq 1$, and $\epsilon \geq 0$ is a small parameter. The functions H_0 and H_1 are assumed to be of class C^{r+1} in their arguments with $r \geq 5$ and they generate the Hamiltonian part of the vector field (1) through the symplectic form ω on the phase space $\mathcal{P}$. The map $\omega^\sharp : T^*\mathbb{R}^{2(n+m+1)} \to \mathbb{R}^{2(n+m+1)}$ appearing in (1) is the inverse of the map $\xi \mapsto \{\omega[x](\xi, \cdot)\}$ with $x \in \mathbb{R}^{2(n+m+1)}$ and $\xi \in T_x\mathbb{R}^{2(n+m+1)}$. The function g is of class C^r and it corresponds to the dissipative part of the perturbation to the unperturbed limit $\epsilon = 0$. We make the following basic assumptions on system (1):

(H1) There exists a $2m-$dimensional manifold $\Pi \subset \mathcal{P}$ which is invariant under the flow of (1) for $\epsilon = 0$. Furthermore, the manifold Π is symplectic, i.e., the restricted two-form

$$\omega_\Pi = \omega|\Pi$$

is nondegenerate.

(H2) For $\epsilon = 0$, system (1) restricted to Π becomes an m-degree-of-freedom, completely integrable Hamiltonian system, i.e., it admits m independent integrals which are in involution with respect to the Poisson bracket induced by the symplectic form ω_Π.

By assumption (H2), the Liouville-Arnold-Jost theorem (see, e.g., Arnold [3]) guarantees the existence of an open set $\mathcal{N} \subset \Pi$ on which we can introduce canonical action-angle variables $(I, \phi) \in \mathbb{R}^m \times \mathbb{T}^m$. (If the level surfaces of H_0 are not compact within the set $\mathcal{N}$, then we have $\phi \in \mathbb{R}^n$, but all of our forthcoming results are still valid.) We assume that the frequency vector $\dot{\phi}$ vanishes on one of these tori, i.e.,

(H3) For $\epsilon = 0$ there exists an m-dimensional torus $\mathcal{C} \subset \mathcal{N}$ given by $I = I_0$ which is completely filled with equilibria of system (1). Furthermore, for any point $p \in \Pi$, the Jacobian $M = D\omega^\sharp H_0(x)|_{x=p}$ admits precisely m pairs of zero eigenvalues, a pair $\pm\lambda_0$ of nonzero real eigenvalues, and n pairs of simple, purely imaginary, nonzero eigenvalues $i\lambda_1, \ldots, i\lambda_n$.

This assumption implies the presence of a stable, an unstable, and $2n$ neutrally stable directions transverse to the manifold Π in the unperturbed limit of system (1). We stress that in (H3) we assumed the eigenvalues and eigenvectors of M to be independent of the point $p \in \mathcal{C}$.

Since the normal bundle of the torus $\mathcal{C}$ is trivial within Π, the independence of stable, unstable and center subspaces of points on $\mathcal{C}$ allows us to introduce local coordinates $y = (y_1, y_2) \in \mathbb{R}^2$ and $z \in \mathbb{R}^{2n}$ in a neighborhood $S_0 \subset \mathcal{P}$ of the set $\mathcal{N}$. The coordinates are such that Eq. (1) can be rewritten in the form

$$\begin{aligned} \dot{y} &= \Lambda y + \bar{Y}(y, z, I, \phi; \epsilon), \\ \dot{z} &= A\underline{z} + \bar{Z}(y, z, I, \phi; \epsilon), \\ \dot{I} &= \epsilon \bar{E}(y, z, I, \phi; \epsilon), \\ \dot{\phi} &= \bar{F}_0(y, z, I, \phi) + \epsilon \bar{F}_\epsilon(y, z, I, \phi; \epsilon). \end{aligned} \tag{2}$$

Here Λ is a diagonal matrix with eigenvalues $\pm\lambda$, and A has the eigenvalues $i\lambda_1, \ldots, i\lambda_n$. Hence there exists a constant $C_A > 0$ such that

$$|e^{At} z| \leq C_A |z|. \tag{3}$$

Note that in the local coordinates we introduced the manifold Π satisfies the equations $y = 0$ and $z = 0$.

Our next major assumption is that

(H4) For $\epsilon = 0$, the torus $\mathcal{C}$ admits a unique, codimension two center manifold

$$\mathcal{M}_0 = \left\{ (y, z, I, \phi) \mid y = y^0(z, I, \phi),\ (z, I, \phi) \in V \subset \mathbb{R}^{2(n+m)} \right\},$$

where the function $y^0(z, I, \phi)$ is of class C^r.

By the uniqueness of this center manifold, $\Pi \subset \mathcal{M}_0$ must hold (at least locally near $\mathcal{C}$), which implies

$$y^0(0, I, \phi) = 0.$$

We note that the existence and uniqueness of $\mathcal{M}_0$ is usually easy to verify if the unperturbed part of system (2) is integrable. In all applications we know of, this integrability is due to the fact that the system is invariant under rotations in ϕ. In such cases the function y^0 has no explicit ϕ-dependence.

Taking V small enough, we can ensure that $\mathcal{M}_0$ is a normally hyperbolic invariant manifold which admits codimension one stable and unstable manifolds of class C^r, denoted $W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$, respectively.

Our next assumption is the existence of a homoclinic structure in the unperturbed problem. In particular, we assume that

(H5) The manifolds $W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$ coincide and form two homoclinic manifolds $W_0^+(\mathcal{M}_0)$ and $W_0^-(\mathcal{M}_0)$.

These homoclinic manifolds are foliated by orbits doubly asymptotic to the center manifold $\mathcal{M}_0$. Based on the applications we are interested in, our next main assumption is that

(H6) Each of the two homoclinic manifolds contains a one-parameter family of heteroclinic orbits that connect points on the torus $\mathcal{C}$. In other words, the torus $\mathcal{C}$ has its own m + 1-dimensional stable and unstable manifolds that form two homoclinic manifolds $W_0^+(\mathcal{C})$ and $W_0^-(\mathcal{C})$. Furthermore, the heteroclinic orbits in both $W_0^+(\mathcal{C})$ and $W_0^-(\mathcal{C})$ connect the same pair of points, i.e., the phase shift vector

$$\Delta x = \lim_{t\to\infty} x^h(t) - x^h(-t) = \begin{pmatrix} \Delta y \\ \Delta z \\ \Delta I \\ \Delta\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lim_{t\to\infty} \phi^h(t) - \phi^h(-t) \end{pmatrix} \qquad (4)$$

is the same for any solution $x^h(t)$ in $W_0^+(\mathcal{C}) \cup W_0^-(\mathcal{C})$.

We would like to ensure that a manifold close to Π survives the perturbation. If $n = 0$, i.e., there are no “oscillatory modes” for the linearized dynamics, then $\Pi \equiv \mathcal{M}_0$ is normally hyperbolic, hence it smoothly perturbs to a nearby invariant manifold. For $n > 0$, however, Π in general does not persist. Motivated by the examples listed in Sect. 1, we then require the perturbation to be such that it preserves Π:

(H7) If $n > 0$, then the manifold Π remains invariant under the flow of system (1) for $\epsilon > 0$.

Based on assumptions (H1)-(H7), we can guarantee the persistence of certain invariant manifolds for $\epsilon > 0$ sufficiently small. The following theorem describes the properties of these manifolds.

Theorem 2.1. *Suppose that assumptions (H1)-(H7) hold. Then there exists $\epsilon_0 > 0$ such that for $0 \le \epsilon < \epsilon_0$ the following are satisfied:*

(i) *There exists a unique, codimension-two, locally invariant manifold $\mathcal{M}_\epsilon$ of class C^r which depends on the parameter ϵ in a C^r fashion. If $n > 0$, then the manifold $\mathcal{M}_\epsilon$ contains the invariant manifold Π which satisfies $y = 0$ and $z = 0$. If $n = 0$, then $\mathcal{M}_0 \equiv \Pi$.*

(ii) *The manifold $\mathcal{M}_\epsilon$ has codimension-one local stable and unstable manifolds $W^s_{loc}(\mathcal{M}_\epsilon)$ and $W^u_{loc}(\mathcal{M}_\epsilon)$ that are of class C^r in the variables (y, z, I, ϕ) and ϵ.*

(iii) *The local unstable manifold $W^u_{\mathrm{loc}}(\mathcal{M}_\epsilon)$ is foliated by a negatively invariant family $\mathcal{F}^u = \cup_{p\in\mathcal{M}_\epsilon} f^u(p)$ of C^r curves $f^u(p)$, i.e., $\mathcal{F}^u = W^u_{\mathrm{loc}}(\mathcal{M}_\epsilon)$ and $F^{-t}(f^u(p)) \subset f^u\left(F^{-t}(p)\right)$ for any $t \ge 0$ and $p \in \mathcal{M}_\epsilon$ (here F^t denotes the flow generated by system (1). Moreover, the fibers $f^u(p)$ are of class C^r in ϵ and p, and $f^u(p) \cap f^u(p') = \emptyset$, unless $p = p'$. Finally, there exist $C_u, \lambda_u > 0$ such that if $q \in f^u(p)$ then*

$$\| F^{-t}(q) - F^{-t}(p) \| < C_u e^{-\lambda_u t},$$

for any $t \ge 0$.

(iv) The local stable manifold $W^s_{\text{loc}}(\mathcal{M}_\epsilon)$ admits a positively invariant foliation $\mathcal{F}^s = \cup_{p\in\mathcal{M}_\epsilon} f^u(p)$ with similar properties.

Proof. The statements of the theorem follow from a direct application of the invariant manifold results of Fenichel [13, 14]. We only note that the uniqueness of the perturbed manifold $\mathcal{M}_\epsilon$ implies $\Pi \subset \mathcal{M}_\epsilon$ in statement (i). □

For simplicity, from now on we will not distinguish between the cases $n = 0$ (i.e., no oscillatory modes for the unperturbed linearized flow near the manifold Π) and $n > 0$. As a result, when we refer to the invariant manifold Π for the perturbed system (1), *we mean $\Pi \equiv \mathcal{M}_\epsilon$ in the case of $n = 0$.*

3. Fenichel Normal Form Near $\mathcal{M}_\epsilon$

In this section we derive a normal form which describes the dynamics of system (1) near the normally hyperbolic invariant manifold $\mathcal{M}_\epsilon$ which exists by Theorem 2.1. The normal form is a specific form of a result of Fenichel [14], or more precisely, of the normal form appearing in Tin [39] (see also Jones and Kopell [23]). Since this construction has appeared in several recent papers, we omit the details of the derivation of the normal form. For a detailed proof, the reader may consult Haller [21].

We first introduce the scaling

$$I = I_0 + \sqrt{\epsilon}\eta, \tag{5}$$

to blow up a neighborhood of the torus of equilibria $\mathcal{C}$. Using the coordinates (y, z, η, ϕ), we obtain the following result.

Lemma 3.1. *There exists $\epsilon_0 > 0$ such that for $0 \le \epsilon < \epsilon_0$, a C^r change of coordinates $\mathcal{T}_\epsilon\colon (y, z, \eta, \phi) \mapsto (w, \zeta, \rho, \psi)$ (with a C^r inverse) defined near the manifold $\mathcal{M}_\epsilon$, which puts system (1) in the form*

$$\begin{aligned}
\dot{w}_1 &= [-\lambda + \langle Y_1, w\rangle + \langle Y_2, \zeta\rangle + \sqrt{\epsilon}Y_3]w_1,\\
\dot{w}_2 &= [\ \ \lambda + \langle Y_4, w\rangle + \langle Y_5, \zeta\rangle + \sqrt{\epsilon}Y_6]w_2,\\
\dot{\zeta} &= A\zeta + (Z_1\zeta)\,\zeta + \sqrt{\epsilon}Z_2\zeta + Z_3w_1w_2,\\
\dot{\rho} &= \sqrt{\epsilon}E,\\
\dot{\psi} &= (F_1\zeta)\,\zeta + \sqrt{\epsilon}F_2 + F_3w_1w_2.
\end{aligned} \tag{6}$$

Here the functions $Y_1, Y_4 : \mathcal{P} \times [0, \epsilon_0] \to \mathbb{R}^2,\ Y_2, Y_5 : \mathcal{P} \times [0, \epsilon_0] \to \mathbb{R}^{2n},\ Y_3, Y_6\colon \mathcal{P} \times [0, \epsilon_0] \to \mathbb{R},\ E, F_2, F_3 : \mathcal{P}\times[0, \epsilon_0] \to \mathbb{R}^m,\ Z_3 : \mathcal{P}\times[0, \epsilon_0] \to \mathbb{R}^{2n}, Z_2 : \mathcal{P}\times[0, \epsilon_0] \to \mathbb{R}$, and the 3-tensors $Z_1\colon \mathcal{P}\times[0, \epsilon_0] \to \mathbb{R}^{2n\times2n\times2n}$ and $F_1\colon \mathcal{P}\times[0, \epsilon_0] \to \mathbb{R}^{m\times2n\times2n}$ are all of class C^{r-4} in their arguments, and $\langle\cdot, \cdot\rangle$ denotes the usual Euclidean inner product. Moreover,

$$D_wZ_1 = 0, \quad D_wZ_2 = 0, \quad D_wF_1 = 0, \quad D_wF_2 = 0. \tag{7}$$

Proof. Based on the references cited above, the proof of this theorem is a routine exercise following the steps outlined in Fenichel [14]. These steps involve changes of coordinates that “straighten out” the manifolds $\mathcal{M}_\epsilon$, $W^s_{\text{loc}}(\mathcal{M}_\epsilon)$, and $W^u_{\text{loc}}(\mathcal{M}_\epsilon)$, as well as their invariant foliations. For a detailed proof we refer the reader to Haller [21]. □

4. Dynamics Near the Manifold $\mathcal{M}_\epsilon$

In this section we use the normal form (6) to study trajectories in a neighborhood of the manifold $\mathcal{M}_\epsilon$. The trajectories of interest lie in the unstable manifold $W^u(\mathcal{M}_\epsilon)$ and do not intersect the local stable manifold $W^s_{\text{loc}}(\mathcal{M}_\epsilon)$ upon entering a small neighborhood of $\mathcal{M}_\epsilon$. Since $\mathcal{M}_\epsilon$ is of "saddle-type", such trajectories pass near the manifold and leave its neighborhood. The question is how the coordinates (w, ζ, ρ, ψ) change during this passage and how the change depends on their initial values upon entry.

By Lemma 3.1, the flow of system (1) near the manifold $\mathcal{M}_\epsilon$ is C^r-conjugate to the flow of the normal form (6) in a neighborhood of the set $w = 0$. In other words, for $\epsilon \leq \epsilon_0$ the normal form is related to the original system within some fixed open set

$$S_0 = \{(w, \zeta, \rho, \psi) \mid |w| < K_w,\ |\zeta| < K_\zeta,\ \sqrt{\epsilon}|\rho| < K_I,\ \psi \in \mathbb{T}^m \},$$

where K_ζ, K_ρ, and K_I are fixed positive constants. We shall primarily be interested in solutions $x(t) = (w(t), \zeta(t), \rho(t), \psi(t))$ of the normal form which enter a small, fixed "box"

$$U_0 = \left\{ (w, \zeta, \rho, \psi) \in S_0 \mid\ |w_i| \leq \delta_0 < \frac{K_w}{4\sqrt{2}},\ |\zeta| \leq \delta_0 < K_\zeta,\ |\rho| \leq K_\rho < \frac{K_I}{\sqrt{\epsilon}} \right\}$$

with positive constants δ_0 and K_ρ. Since the functions on the right-hand-side of (6) are of class C^{r-4}, on the closure of S_0 they obey the estimates

$$\begin{aligned} |Y_i|,\ |Z_j|,\ |E|,\ |F_k| &< B_0, \\ |DY_i|,\ |DZ_j|,\ |DE|,\ |DF_k| &< B_0, \end{aligned} \tag{8}$$

for all $0 \leq \epsilon \leq \epsilon_0$ and for appropriate $B_0 > 0$. We want to follow a solution $x(t)$ which enters the set U_0 by intersecting its boundary ∂U_0 within the domain

$$\partial_1 U_0 = \{(w, \zeta, \rho, \psi) \in \partial U_0 \mid |\zeta| < \delta_0,\ |\rho| \leq K_\rho\}$$

at time $t = 0$. For such a solution we have $w_1(0) = \delta_0$, and we assume that for $0 < \epsilon \leq \epsilon_0$, the rest of the coordinates of the entry point $x(0)$ obey the *entry conditions*

$$|\zeta(0)| < c_1\epsilon^\beta, \quad \frac{c_2\epsilon}{\delta_0} < |w_2(0)| < \frac{c_3\epsilon}{\delta_0}, \quad |\rho(0)| < c_4 < K_\rho \tag{9}$$

for fixed positive constants $c_1, \ldots, c_4$ and for some power $\frac{1}{2} < \beta < 1$.

The second inequality in (9) implies that the solution $x(t)$ enters U_0 close to the local stable manifold $W^s_{\text{loc}}(\mathcal{M}_\epsilon)$. Such solutions spend a long time within U_0, and hence their $\zeta(t)$ component does not necessarly remain under control on such time scales, i.e., $x(t)$ may not exit U_0 through the domain $\partial_1 U_0$ of its boundary. An exit through $\partial_1 U_0$ means that $|\zeta|$ remains bounded by δ_0 while $x(t)$ is in U_0. Our first result shows that this is indeed the case.

Lemma 4.1. *Suppose that for a solution $x(t)$, the entry conditions in (9) are satisfied. Then for any fixed constant β with $\frac{1}{2} < \beta < 1$, there exist $\epsilon_1 > 0$ and $\delta_1 > 0$ such that for all $0 < \delta_0 < \delta_1$ and $0 < \epsilon_0 < \epsilon_1$ there exists $T^* > 0$ with $x(T^*) \in \partial_1 U_0$. Moreover, the minimal such time T^* obeys the estimate*

$$T^* < T_\epsilon = \frac{2}{\lambda} \log \frac{\delta_0^2}{c_2\epsilon}. \tag{10}$$

Proof. We start by picking constants B_ζ and α with $B_\zeta > c_1 > 0$ and $\beta < \alpha < 1$. Then, by the smoothness of the solution $\underline{x}(t)$ with respect to t, (9) implies the existence of a time $\bar{T} > 0$ such that for all $t \in [0, \bar{T})$, we have

$$|\zeta(t)| \le B_\zeta \epsilon^\beta, \qquad |\rho(t)| \le K_\rho, \qquad |w_1(t)w_2(t)| \le \frac{c_3}{\delta_0}\epsilon^\alpha. \tag{11}$$

Clearly, for ϵ small enough, (11) implies $x(t) \in S_0$. By the continuity of $x(t)$ in t, we also have $x(t) \in U_0$ for $t > 0$ small enough. It is also clear that $\bar{T}$ can be slightly increased so that the inequalities above still hold. Let $T^* > 0$ denote the time when $x(t)$ first intersects the boundary ∂U_0. One can easily check that $T^* < T_\epsilon$ must hold by assuming the contrary and observing that such an assumption would lead to $|w_2(T_\epsilon)| > |w_{20}| \exp\left(\lambda T^*/2\right) > \delta_0$, which is a contradiction. We want to argue that $\bar{T}$ can in fact be increased up to T^*.

Let us assume that for all fixed B_ζ, K_ρ, and α, there exists a time T_0 with $\bar{T} \le T_0 < T_\epsilon$ such that (11) holds for all $t < T_0$, but at least one of the inequalities is violated at $t = T_0$. We will consider these inequalities individually and argue that none of them can be violated at $t = T_0$ if we choose ϵ and δ_0 small enough and select B_ζ, K_ρ, and α properly. We note that $|w_2| < \sqrt{2}\delta_0$ will automatically hold in our argument since T_0 is smaller than the exit time T^*.

By assumption, the third equation of (6) yields the following estimate for all $0 \le t < T_0$ on the solution $x(t)$:

$$\begin{aligned} |\zeta(t)| &= |e^{At}\zeta(0)| + \int_0^t |e^{A(t-s)}\left((Z_1\zeta)\,\zeta + \sqrt{\epsilon}Z_2\zeta + Z_3w_1w_2\right)|\, ds \\ &< C_A|\zeta(0)| + C_AB_0\int_0^t (B_\zeta\epsilon^\beta|\zeta(s)| + \sqrt{\epsilon}|\zeta(s)| + \tfrac{c_3}{\delta_0}\epsilon^\alpha)\, ds \\ &< C_A[c_1\epsilon^\beta + B_0\tfrac{c_3}{\delta_0}\epsilon^\alpha T_\epsilon] + 2C_AB_0B_\zeta\sqrt{\epsilon}\int_0^t |\zeta(s)|\, ds, \end{aligned}$$

where we used (3). By the Gronwall inequality, this implies

$$|\zeta(t)| = C_A[c_1\epsilon^\beta + B_0\frac{c_3}{\delta_0}\epsilon^\alpha T_\epsilon]\; e^{2C_AB_0B_\zeta\sqrt{\epsilon}T_\epsilon} < 2ec_1C_A\epsilon^\beta \tag{12}$$

for $\epsilon > 0$ small. Since (12) holds for all $0 \le t < T_0$, by the continuity of $|\zeta(t)|$, we obtain

$$|\zeta(T_0)| < 2ec_1C_A\epsilon^\beta < B_\zeta\epsilon^\beta, \tag{13}$$

if we choose $B_\zeta = 7c_1C_A$. Therefore, the first inequality in (11) cannot be violated at $t = T_0$. We now study the second inequality in (11).

Using the fourth equation in (6), for $0 \le t < T_0$ we can estimate the ρ-component of the solution $x(t)$ as

$$|\rho(t)| < |\rho(0)| + \sqrt{\epsilon}\int_0^t |E|\, ds < |\rho(0)| + \sqrt{\epsilon}B_0t < c_4 + \frac{2B_0}{\lambda}\sqrt{\epsilon}\log\frac{\delta_0^2}{c_2\epsilon} < c_4 + 1, \tag{14}$$

for small ϵ. Thus, selecting $K_\rho = c_4 + 2$ and using the continuity of the function $\rho(t)$, we obtain from (14) that the second inequality in (11) cannot be violated at $t = T_0$ either.

As far as the last inequality in (11), the normal form (6) yields the differential equation

$$\frac{d}{dt}(w_1w_2) = [\langle Y_1 + Y_{4,}\, w\rangle + \langle Y_2 + Y_{5,}\, \zeta\rangle + \sqrt{\epsilon}(Y_3 + Y_6)]w_1w_2. \tag{15}$$

From this equation we obtain that for $0 \leq t < T_0$, the product of the two w-components of the solution $x(t)$ admits the estimate

$$\begin{aligned} |w_1(t)w_2(t)| \leq |w_1(0)w_2(0)| + \int_0^t \Big| \langle Y_1 + Y_{4,}\ w\rangle \\ + \langle Y_2 + Y_{5,}\ \zeta\rangle + \sqrt{\epsilon}(Y_3 + Y_6)\Big| |w_1(s)w_2(s)|\ ds \\ < c_3\epsilon + \int_0^t 2B_0[\sqrt{2}\delta_0 + B_\zeta\epsilon^\beta + \sqrt{\epsilon}]|w_1(s)w_2(s)|\ ds. \end{aligned}$$

Then a simple Gronwall estimate shows that

$$|w_1(t)w_2(t)| \leq c_3\epsilon \exp\left\{2B_0[2\sqrt{2}\delta_0 + B_\zeta\epsilon^\beta + \sqrt{\epsilon}]T_\epsilon\right\} < \frac{c_3}{\delta_0}\epsilon \exp 4B_0[\sqrt{2} + 1]T_\epsilon,$$

which implies that

$$|w_1(t)w_2(t)| < c_3 \left(\frac{\delta_0^2}{c_2}\right)^{4B_0[\sqrt{2}+1]\frac{\delta_0}{\lambda}} \epsilon^{1-8B_0[\sqrt{2}+1]\frac{\delta_0}{\lambda}} < c_3\epsilon^\alpha, \tag{16}$$

if we choose δ_0 small enough such that

$$\left(\frac{\delta_0^2}{c_2}\right)^{4B_0[\sqrt{2}+1]\frac{\delta_0}{\lambda}} < 1, \qquad \delta_0 < \frac{\lambda(1-\alpha)}{4B_0(1+\sqrt{2})}$$

hold. Again, by continuity with respect to t, (16) implies $|w_1(T_0)w_2(T_0)| \leq c_3\epsilon^\alpha/\delta_0$, hence the last inequality in (11) cannot be violated at $t = T_0$ either. But this contradicts our original assumption on the time T_0 and proves the statement of the lemma. □

In the following lemma we describe how the coordinates of passing trajectories change and how this change depends on the initial values of these coordinates upon entry into the neighborhood U_0.

Lemma 4.2. *Let us fix a constant $\frac{1}{2} < \beta < 1$ and assume that for $0 < \epsilon < \epsilon_0$ and $\delta_0 < \delta_1$, the entry conditions (9) hold for a solution $x(t)$ which enters the set U_0 at $t = 0$ and leaves it at $t = T^*$. Let us introduce the notation $a = (w_{20}, \zeta_0, \rho_0, \psi_0)$ and let $x_0 = (\delta_{0,} a)$ and $x^* = x(T^*) = (w_1^*, \delta_0, \zeta^*, \rho^*, \psi^*)$ define the coordinates of the solution at entry and departure, respectively. Then there exist constants $K > 0$, $0 < \mu < \frac{1}{2}$, and $\delta_0^* > 0$, and for any $\delta_0 < \delta_0^*$ there exists $\epsilon_0^* > 0$ such that for all $0 < \epsilon < \epsilon_0^*$ the following estimates hold:*

(i)

$$|w_1^*| < K\epsilon^\beta,\ |\zeta^* - \zeta_0| < K\epsilon^\beta,\ |\rho^* - \rho_0| < K\sqrt{\epsilon}^\beta,\ |\psi^* - \psi_0| < K\sqrt{\epsilon}^\beta.$$

(ii)

$$\begin{aligned} &|D_a\, w_1^*| < K\epsilon^\beta, && |D_a\, \zeta^* - (0,1,0,0)| < K\epsilon^\mu, \\ &|D_a\rho^* - (0,0,1,0)| < K\epsilon^\mu, && |D_a\, \psi^* - (0,0,0,1)| < K\epsilon^\mu. \end{aligned}$$

(iii)

$$|D_{\epsilon^\mu}\, w_1^*| < K\epsilon^\beta, \quad |D_{\epsilon^\mu}\zeta^*| < K\epsilon^\mu, \quad |D_{\epsilon^\mu}\, \rho^*| < K\epsilon^\mu, \quad |D_{\epsilon^\mu}\psi^*| < K\epsilon^\mu.$$

Proof. We start the proof by establishing a lower estimate and a refined upper estimate for the exit time T^*. From the normal form (6) we easily obtain that

$$|w_{20}|\, e^{(\lambda+3\delta_0 B_0)t} > |w_2(t)| > |w_{20}|\, e^{(\lambda-3\delta_0 B_0)t}, \tag{17}$$

which in turn gives

$$T_1 = \frac{1}{\lambda + 3\delta_0 B_0} \log \frac{\delta_0^2}{c_2\epsilon} < T^* < T_2 = \frac{1}{\lambda - 3\delta_0 B_0} \log \frac{\delta_0^2}{c_2\epsilon} \tag{18}$$

for any solution with initial conditions satisfying the estimates in (9).

We now turn to the proof of statement (i). From (6) we obtain that

$$|w_1^*| = |w_1(T^*)| < |w_1(T_1)| < |w_{10}|\, e^{-(\lambda-3\delta_0 B_0)T_1} < \delta_0 \left(\frac{\delta_0^2}{c_2}\right)^{\frac{\lambda-3\delta_0 B_0}{\lambda+3\delta_0 B_0}} \epsilon^{\beta} \tag{19}$$

provided

$$\delta_0 < \frac{\lambda(1-\beta)}{3B_0(1+\beta)}. \tag{20}$$

By Lemma 4.1, all inequalities in (9) hold for $t \in [0, T^*]$, thus selecting $B_\zeta = 7c_1 C_A$ (as in the proof of that lemma) and setting $t = T^*$, we obtain

$$|\zeta^*| < B_\zeta \epsilon^\beta.$$

This inequality and (9) imply that

$$|\zeta^* - \zeta_0| \le |\zeta^*| + |\zeta_0| < (B_\zeta + 1)\epsilon^\beta. \tag{21}$$

From the third equation in (6) we see that

$$|\rho^* - \rho_0| \le \sqrt{\epsilon} \int_0^{T^*} |E|_{x(t)}\, dt < \sqrt{\epsilon} B_0 T_\epsilon < \frac{2B_0}{\lambda}\sqrt{\epsilon} \log \frac{\delta_0^2}{c_2\epsilon} < \frac{2B_0}{\lambda}\sqrt{\epsilon}^{\beta}. \tag{22}$$

Finally, the last equation in (6) and (11) yield the estimate

$$\begin{aligned} |\psi^* - \psi_0| &\le \int_0^{T^*} \left[|(F_1\zeta)\,\zeta| + \sqrt{\epsilon}\, |F_2| + |F_3|\, |w_1 w_2| \right]_{x(t)}\, dt \\ &< \left[B_\zeta^2 B_0 \epsilon^{2\beta} + \sqrt{\epsilon} B_0 + \tfrac{B_0 c_3}{\delta_0} \epsilon^\alpha \right] T_\epsilon \\ &< \tfrac{2B_0}{\lambda} \left[B_\zeta^2 + \tfrac{c_3}{\delta_0} + 1 \right] \sqrt{\epsilon}^{\beta}. \end{aligned} \tag{23}$$

But then (19), (21), (22), and (23) show that statement (i) of the lemma is satisfied if we choose $K > 0$ big enough.

To prove statement (ii), we first need the variational equation associated with the normal form (6). We shall only sketch the proof of the estimates in (ii) for the derivatives of x^* with respect to ρ_0. To this end, we need the ρ_0-variational equation associated with the normal form (6):

$$
\begin{aligned}
\tfrac{d}{dt}\left(D_{\rho_0} w_1\right) &= \left[-\lambda + \langle Y_1, w\rangle + \langle Y_2, \zeta\rangle + \sqrt{\epsilon} Y_3\right] D_{\rho_0} w_1 \\
&\quad + \left[\langle DY_1 D_{\rho_0} x, w\rangle + \langle Y_1, D_{\rho_0} w\rangle\right. \\
&\quad \left. + \langle DY_2 D_{\rho_0} x, \zeta\rangle + \langle Y_2, D_{\rho_0}\zeta\rangle + \sqrt{\epsilon} DY_3 D_{\rho_0} x\right] w_1, \\
\tfrac{d}{dt}\left(D_{\rho_0} w_2\right) &= \left[\ \ \lambda + \langle Y_4, w\rangle + \langle Y_5, \zeta\rangle + \sqrt{\epsilon} Y_6\right] D_{\rho_0} w_2 \\
&\quad + \left[\langle DY_4 D_{\rho_0} x, w\rangle + \langle Y_4, D_{\rho_0} w\rangle\right. \\
&\quad \left. + \langle DY_5 D_{\rho_0} x, \zeta\rangle + \langle Y_5, D_{\rho_0}\zeta\rangle + \sqrt{\epsilon} DY_6 D_{\rho_0} x\right] w_2, \\
\tfrac{d}{dt}\left(D_{\rho_0}\zeta\right) &= A D_{\rho_0}\zeta + \left(DZ_1 D_{\rho_0} x\, \zeta\right)\zeta + (Z_1\zeta) D_{\rho_0}\zeta \\
&\quad + \left(Z_1 D_{\rho_0}\zeta\right)\zeta + \sqrt{\epsilon}\,\langle DZ_2 D_{\rho_0} x,\ \zeta\rangle \\
&\quad \sqrt{\epsilon}\,\langle Z_2,\ D_{\rho_0}\ \zeta\rangle + DZ_3 D_{\rho_0} x w_1 w_2 + Z_3 D_{\rho_0}(w_1 w_2), \\
\tfrac{d}{dt}\left(D_{\rho_0}\rho\right) &= \sqrt{\epsilon} DE D_{\rho_0} x, \\
\tfrac{d}{dt}\left(D_{\rho_0}\psi\right) &= \left(DF_1 D_{\rho_0} x\, \zeta\right)\zeta + \left(F_1 D_{\rho_0}\zeta\right)\zeta + (F_1\zeta) D_{\rho_0}\zeta \\
&\quad + \sqrt{\epsilon}\,\langle DF_2, D_{\rho_0} x\rangle + \langle DF_3, D_{\rho_0} x\rangle w_1 w_2 \\
&\quad + F_3 D_{\rho_0}(w_1 w_2).
\end{aligned}
\tag{24}
$$

Let us select constants α, γ, μ, and ν with

$$
0 < \mu < \nu < \frac{1}{2} < \gamma < \beta < \alpha < 1. \tag{25}
$$

Then, by the smoothness of the solution $x(t)$ with respect to t, there exists a time $T_0 \le T^*$ such that for all $t \in [0, T_0)$ and for $\epsilon > 0$ sufficiently small,

$$
|D_{\rho_0}\zeta(t)| \le B_\zeta^{'}\epsilon^{\gamma}, \quad |D_{\rho_0}\rho(t) - 1| \le K_\rho^{'}\epsilon^{\mu}, \quad |D_{\rho_0}\psi(t)| \le K_\psi^{'}\epsilon^{\mu}, \tag{26}
$$

$$
\begin{aligned}
&|D_{\rho_0}[w_1(t) w_2(t)]| \le K_0^{'}\epsilon^{\beta}, \\
&|D_{\rho_0} w_1(t)| \le K_w^{'}\epsilon^{\beta},\ |D_{\rho_0} w_2(t)| \le K_w^{'}\epsilon^{-\nu},\ \|D_{\rho_0} x(t)\| \le 2K_w^{'}\epsilon^{-\nu},
\end{aligned}
\tag{27}
$$

with appropriate positive constants $B_\zeta^{'}, K_\rho^{'}, K_\psi^{'}, K_0^{'}$, and $K_w^{'}$. We also recall that for $t \in [0, T^*]$, the inequalities in (11) hold, and we have $T^* < T_\epsilon$ by Lemma 4.1.

As in the proof of Lemma 4.1, we shall argue that none of the inequalities in (26) and (27) can be violated at $t = T_0$ if we choose the constants appearing in those inequalities properly. Thus we can select $T_0 = T^*$, i.e., we obtain estimates of the form (26) and (27) on the whole time interval while the solution $x(t)$ stays inside the set U_0. We shall use these estimates to prove statement (ii) of the lemma.

We start by considering the inequalities in (26). From the third equation in (24) we obtain that

$$
\begin{aligned}
|D_{\rho_0}\zeta(t)| \le &\int_0^t C_A\left[B_\zeta^2\epsilon^{2\beta} B_0 2K_w^{'}\epsilon^{-\nu} + 2B_\zeta B_0\epsilon^{\beta}|D_{\rho_0}\zeta(t)|\right] \\
&+ C_A\left[\sqrt{\epsilon} B_0 2K_w^{'}\epsilon^{-\nu} B_\zeta\epsilon^{\beta} + \sqrt{\epsilon} B_0|D_{\rho_0}\zeta(t)| + B_0 2K_w^{'}\epsilon^{-\nu}\frac{c_3}{\delta_0}\epsilon^{\alpha}\right]\ ds \\
< &\ 2C_A K_w^{'} B_0\left[B_\zeta^2\epsilon^{2\beta-\nu} + B_\zeta\epsilon^{\frac{1}{2}+\beta-\nu} + \frac{c_3}{\delta_0}\epsilon^{\alpha-\nu}\right] T_0
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t 2C_A B_\zeta B_0 \epsilon^\beta |D_{\rho_0}\zeta(t)| \; ds \qquad (28)\\
&< \frac{1}{e} B_\zeta^{'} \epsilon^\gamma + \int_0^t 2C_A B_\zeta B_0 \epsilon^\beta |D_{\rho_0}\zeta(t)| \; ds,
\end{aligned}
$$

provided we choose ν small enough so that

$$\alpha - \gamma > \nu, \tag{29}$$

and select ϵ small enough. Then the Gronwall inequality applied to (29) shows that for all $t \in [0, T_0]$,

$$|D_{\rho_0}\zeta(t)| \leq B_\zeta^{'} \epsilon^\gamma \; \exp\left[\frac{4\sqrt{\epsilon} C_A B_0 B_z}{\lambda} \log \frac{\delta_0^2}{c_2 \epsilon}\right] \leq B_\zeta^{'} \epsilon^\gamma, \tag{30}$$

for ϵ small. For all $t \in [0, T_0]$, from the fourth equation in (24) we obtain the estimate

$$|D_{\rho_0}\rho(t) - 1| \leq K_\rho^{'} \epsilon^\mu,$$

if we select μ small enough such that

$$\frac{1}{2} - \nu > \mu. \tag{31}$$

Using the last equation in (24), we see that for all $t \in [0, T_0]$,

$$
\begin{aligned}
|D_{\rho_0}\psi(t)| &\leq \textstyle\int_0^t (B_0 2K_w^{'} \epsilon^{-\nu} B_\zeta^2 \epsilon^{2\beta} + B_0 B_\zeta^{'} \epsilon^\gamma B_\zeta \epsilon^\beta + B_0 B_\zeta \epsilon^\beta B_\zeta^{'} \epsilon^\gamma + \sqrt{\epsilon} B_0 2K_w^{'} \epsilon^{-\nu} \\
&\quad + B_0 2K_w^{'} \epsilon^{-\nu} \tfrac{c_3}{\delta_0} \epsilon^\alpha + B_0 K_0^{'} \epsilon^\beta) \; ds \\
&< B_0 \left[2B_\zeta^2 K_w^{'} \epsilon^{2\beta-\nu} + 2B_\zeta B_\zeta^{'} \epsilon^{\beta+\gamma} + 2K_w^{'} \epsilon^{\frac{1}{2}-\nu} + 2K_w^{'} \tfrac{c_3}{\delta_0} \epsilon^{\alpha-\nu} + K_0^{'} \epsilon^\beta\right] T_\epsilon \\
&< K_\psi^{'} \epsilon^\mu,
\end{aligned} \tag{32}
$$

provided (29) and (31) hold.

To estimate the time interval on which the last inequality in (26) holds, we note that the time evolution of the quantity $w_2 D_{\rho_0} w_1$ is given by the equation

$$
\begin{aligned}
\tfrac{d}{dt}\left(w_2 D_{\rho_0} w_1\right) = \; &[\langle DY_1 D_{\rho_0} x, \; w\rangle + \langle Y_1, D_{\rho_0} w\rangle + \langle DY_2 D_{\rho_0} x, \; \zeta\rangle + \langle Y_2, \; D_{\rho_0}\zeta\rangle \\
&+ \sqrt{\epsilon} \, \langle DY_{3,} \, D_{\rho_0} x\rangle] w_1 w_2 \\
&+ \left[\langle Y_1 + Y_4, \; w\rangle + \langle Y_2 + Y_5, \; \zeta\rangle + \sqrt{\epsilon}\,(Y_3 + Y_6)\right] w_2 D_{\rho_0} w_1.
\end{aligned} \tag{33}
$$

We now estimate the terms on the right-hand-side of this expression individually on the time interval $[0, T_0)$. The first term can be estimated as

$$
\begin{aligned}
&|\langle DY_1 D_{\rho_0} x, \; w\rangle \, w_1 w_2| < B_0 \left(w_1^2 \, |w_2| + w_2^2 \, |w_1|\right) \\
&\quad \left(|D_{\rho_0} w_1| + |D_{\rho_0} w_2| + |D_{\rho_0}\zeta| + |D_{\rho_0}\rho| + |D_{\rho_0}\psi|\right) \\
&< B_0 \left[|w_2 D_{\rho_0} w_1| \left(w_1^2 + |w_1 w_2|\right) + |w_1 D_{\rho_0} w_2| \left(w_2^2 + |w_1 w_2|\right)\right. \\
&\quad \left. + 3 \, |D_{\rho_0}\rho| \, |w_1 w_2| \left(|w_1| + |w_2|\right)\right] \\
&< 2\delta_0^2 B_0 \left(|w_2 D_{\rho_0} w_1| + |w_1 D_{\rho_0} w_2|\right) + 12 B_0 c_3 \epsilon^\alpha.
\end{aligned} \tag{34}
$$

In a similar fashion, we can estimate the remaining terms to obtain

$$
\begin{aligned}
\left|\langle Y_1, D_{\rho_0} w\rangle\, w_1 w_2\right| &< \delta_0 B_0 \left(|w_2 D_{\rho_0} w_1| + |w_1 D_{\rho_0} w_2|\right), \\
\left|\langle DY_2 D_{\rho_0} x,\, \zeta\rangle\, w_1 w_2\right| &< \delta_0 B_\zeta B_0 \epsilon^\beta \left(|w_2 D_{\rho_0} w_1| + |w_1 D_{\rho_0} w_2| + \tfrac{6c_3}{\delta_0^2}\epsilon^\alpha\right), \\
\left|\langle Y_{2,}\, D_{\rho_0}\zeta\rangle\, w_1 w_2\right| &< B_0 \tfrac{c_3 B_\zeta^{'}}{\delta_0}\epsilon^{\alpha+\gamma}, \\
\sqrt{\epsilon}\left|\langle DY_{3,}\; D_{\rho_0} x\rangle\, w_1 w_2\right| &< \delta_0 B_0 \sqrt{\epsilon}\left(|w_2 D_{\rho_0} w_1| + |w_1 D_{\rho_0} w_2| + \tfrac{6c_3}{\delta_0^2}\epsilon^\alpha\right), \\
\left|\langle Y_1 + Y_4,\, w\rangle\, w_2 D_{\rho_0} w_1\right| &< 2\delta_0 B_0 \left|w_2 D_{\rho_0} w_1\right|, \\
\left|\langle Y_2 + Y_5,\, \zeta\rangle\, w_2 D_{\rho_0} w_1\right| &< 2B_\zeta B_0 \epsilon^\beta \left|w_2 D_{\rho_0} w_1\right|, \\
\left|(Y_3 + Y_6)\, w_2 D_{\rho_0} w_1\right| &< 2B_0 \sqrt{\epsilon}\left|w_2 D_{\rho_0} w_1\right|.
\end{aligned}
\tag{35}
$$

Integrating (33) and using the estimates (34)–(35), we find that for all $t \in [0, T_0)$,

$$
\begin{aligned}
&|w_2(t) D_{\rho_0} w_1(t)| \\
&< \int_0^t \delta_0 B_0 \left[11\,|w_2 D_{\rho_0} w_1| + 5\,|w_1 D_{\rho_0} w_2|\right] + \frac{c_3 B_0}{\delta_0}\left(B_\zeta^{'} + 12\delta_0 + 6(B_\zeta + 1)\right)\epsilon^\alpha\, ds,
\end{aligned}
$$

which gives

$$
\begin{aligned}
|w_2(t) D_{\rho_0} w_1(t)| < \tfrac{2c_3 B_0}{\delta_0 \lambda}\left(B_\zeta^{'} + 12\delta_0 + 6(B_\zeta + 1)\right)\epsilon^\alpha \log \tfrac{\delta_0^2}{c_2 \epsilon} \\
+ \textstyle\int_0^t \delta_0 B_0 \left[11\,|w_2 D_{\rho_0} w_1| + 5\,|w_1 D_{\rho_0} w_2|\right]\; ds.
\end{aligned}
\tag{36}
$$

By the symmetry of the normal form (6), we immediately obtain

$$
\begin{aligned}
|w_1(t) D_{\rho_0} w_2(t)| < \tfrac{2c_3 B_0}{\delta_0 \lambda}\left(B_\zeta^{'} + 12\delta_0 + 6(B_\zeta + 1)\right)\epsilon^\alpha \log \tfrac{\delta_0^2}{c_2 \epsilon} \\
+ \textstyle\int_0^t \delta_0 B_0 \left[11\,|w_1 D_{\rho_0} w_2| + 5\,|w_2 D_{\rho_0} w_1|\right]\; ds.
\end{aligned}
\tag{37}
$$

Adding the two inequalities (36) and (37), then applying a Gronwall estimate to the resulting inequality, we obtain

$$
\begin{aligned}
|w_2(t) D_{\rho_0} w_1(t)| + |w_1(t) D_{\rho_0} w_2(t)| &< \left[\tfrac{4c_3 B_0}{\delta_0 \lambda}\left(B_\zeta^{'} + 12\delta_0 + 6(B_\zeta + 1)\right)\epsilon^\alpha \log \tfrac{\delta_0^2}{c_2\epsilon}\right] \\
&\quad \times \exp\left(\tfrac{32\delta_0 B_0}{\lambda}\log \tfrac{\delta_0^2}{c_2\epsilon}\right) \\
&< K_0^{'}\epsilon^\beta,
\end{aligned}
\tag{38}
$$

where we selected

$$
\alpha - \beta > \frac{32\delta_0 B_0}{\lambda}, \tag{39}
$$

and assumed that ϵ is small enough. Then the inequality (38) implies that for all $t \in [0, T_0]$,

$$
|D_{\rho_0}[w_1(t) w_2(t)]| \le K_0^{'}\epsilon^\beta. \tag{40}
$$

It then remains to verify the last three inequalities in (26) for $t \le T_0$. Using the second inequality in (9) with (17) and (38) yields the estimate

$$
|D_{\rho_0} w_1(t)| < \frac{4c_3 B_0}{c_2 \lambda}\left(B_\zeta^{'} + 12\delta_0 + 6(B_\zeta + 1)\right)\left(\frac{\delta_0^2}{c_2}\right)^{\frac{19\delta_0 B_0 - \lambda}{\lambda - 3\delta_0 B_0}} \epsilon^{\alpha + \frac{16\delta_0 + B_0}{\lambda - 3\delta_0 B_0}}. \tag{41}
$$

If we use (39) and select

$$\delta_0 < \frac{\lambda}{6B_0}, \tag{42}$$

then we obtain

$$|D_{\rho_0} w_1(t)| < K_w' \epsilon^\beta,$$

since $32\delta_0 B_0/\lambda > 16\delta_0 B_0/(\lambda - 3\delta_0 B_0)$. Furthermore, from (36)–(37) we obtain

$$|w_1(t) D_{\rho_0} w_2(t)| < \frac{\bar{K}}{\delta_0} \epsilon^\alpha \log \frac{\delta_0^2}{c_2 \epsilon} e^{16\delta_0 B_0 t}$$

for an appropriate constant $\bar{K}$. This, combined with the easy estimate $|w_1(t)| > \delta_0 \exp[-(\lambda + 3\delta_0 B_0)t]$ from the normal form (6), implies that

$$|D_{\rho_0} w_2(t)| < \frac{\bar{K}}{\delta_0} \epsilon^\alpha \log \frac{\delta_0^2}{c_2 \epsilon} e^{(\lambda + 19\delta_0 B_0)T_2} < \bar{K} \epsilon^{-\nu}, \tag{43}$$

if we choose

$$\alpha + \nu > \frac{\lambda + 19\delta_0 B_0}{\lambda - 3\delta_0 B_0}, \tag{44}$$

and let $\epsilon > 0$ be small enough. Since the last inequality in (27) trivially follows from (26), (41), (43), we conclude from (30)-(32) and (40)-(43) that the estimates in (27) hold for all $t \in [0, T^*]$, provided we satisfy (20),(29), (31), (39), (42), (44), and select ϵ small enough.

We now use (26) and (27) to prove statement (ii) of the lemma. First note that for any initial value $x_0 \in \partial_1 U_0$, the time $t = T^*$ that the corresponding solution $x(t; x_0)$ spends within U_0 is the solution of the equation

$$w_2(t; x_0) = \delta_0, \qquad t \geq 0. \tag{45}$$

From the second equation in (6) we can estimate the magnitude of $\dot{w}_2(T^*)$ as

$$|\dot{w}_2(T^*)| \geq \frac{\lambda}{2} |w_2(T^*)| = \frac{\lambda}{2} \delta_0.$$

This inequality shows that

$$\frac{\partial}{\partial t} w_2 (t; x_0)_{t=T^*} = \dot{w}_2(T^*) \neq 0,$$

hence, by the implicit function theorem, we can solve (45) near (T^*, x_0) to obtain a continuous function $T^*(x_0)$. Moreover, this function is in fact of class C^r, since the solution $w_2(t; x_0)$ is a C^r function of the initial data and depends on t in a C^r fashion. Consequently, the function

$$x^*(x_0) = x(T^*(x_0); x_0)$$

is of class C^r. Using this expression, the derivatives of the components of x^* with respect to ρ_0 can be computed as

$$
\begin{aligned}
D_{\rho_0} w_1^*(x_0) &= -\frac{\dot{w}_1(T^*;x_0)}{\dot{w}_2(T^*;x_0)} D_{\rho_0} w_2(T^*;x_0) + D_{\rho_0} w_1(T^*;x_0), \\
D_{\rho_0} \zeta^*(x_0) &= -\frac{\dot{\zeta}(T^*;x_0)}{\dot{w}_2(T^*;x_0)} D_{\rho_0} w_2(T^*;x_0) + D_{\rho_0} \zeta(T^*;x_0), \\
D_{\rho_0} \rho^*(x_0) &= -\frac{\dot{\rho}(T^*;x_0)}{\dot{w}_2(T^*;x_0)} D_{\rho_0} w_2(T^*;x_0) + D_{\rho_0} \rho(T^*;x_0), \\
D_{\rho_0} \psi^*(x_0) &= -\frac{\dot{\psi}(T^*;x_0)}{\dot{w}_2(T^*;x_0)} D_{\rho_0} w_2(T^*;x_0) + D_{\rho_0} \psi(T^*;x_0),
\end{aligned} \tag{46}
$$

where we used (45). Then, using the normal form (6), the estimates in (26)-(27) with $t = T^*$, and the inequality (31), we obtain from (46) the following estimates:

$$
\begin{aligned}
|D_{\rho_0} w_1^*(x_0)| &< \frac{3\delta_0^2}{c_2\epsilon} e^{-2(\lambda-3\delta_0 B_0)T_1} K_w' \epsilon^{-\nu} + K_w' \epsilon^{\beta} \\
&< K_1 \epsilon^{\frac{\lambda-9\delta_0 B_0}{\lambda+3\delta_0 B_0}-\nu} + K_w' \epsilon^{\beta} < (K_1 + K_w')\epsilon^{\beta}, \\
|D_{\rho_0} \zeta^*(x_0)| &< \frac{2K_w'}{\lambda\delta_0} \left[\|A\| B_\zeta \epsilon^{\beta} + B_\zeta^2 B_0 \epsilon^{2\beta} + \sqrt{\epsilon} B_0 B_\zeta \epsilon^{\beta} + B_0 \frac{c_3}{\delta_0} \epsilon^{\alpha} \right] \epsilon^{-\nu} + \frac{2B_\zeta'}{\lambda\delta_0} \epsilon^{\gamma} \\
&< \left[B_\zeta' + \frac{2K_w'}{\lambda\delta_0} \left(B_0 \left(B_\zeta^2 + B_\zeta + \frac{c_3}{\delta_0} \right) + \|A\| B_\zeta \right) \right] \epsilon^{\mu}, \\
|D_{\rho_0} \rho^*(x_0) - 1| &< \sqrt{\epsilon} \frac{2B_0 K_w'}{\lambda\delta_0} \epsilon^{-\nu} + K_\rho' \epsilon^{\mu} < \left[K_\rho' + \frac{2B_0 K_w'}{\lambda\delta_0} \right] \epsilon^{\mu}, \\
|D_{\rho_0} \psi^*(x_0)| &< \frac{2}{\lambda\delta_0} \left[B_\zeta^2 B_0 \epsilon^{2\beta} + \sqrt{\epsilon} B_0 + B_0 \frac{c_3}{\delta_0} \epsilon^{\alpha} \right] K_w' \epsilon^{-\nu} + K_\psi' \epsilon^{\mu} \\
&< \left[K_\psi' + \frac{2K_w' B_0}{\lambda\delta_0} \left(B_\zeta^2 + 1 + \frac{c_3}{\delta_0} \right) \right] \epsilon^{\mu},
\end{aligned} \tag{47}
$$

if we let

$$
\beta + \nu < \frac{\lambda - 9\delta_0 B_0}{\lambda + 3\delta_0 B_0}. \tag{48}
$$

But (47), together with identical estimates for the rest of the components of $D_a x^*$, implies the inequalities in statement (ii) of the lemma.

It remains to show that the constants we introduced in the proof of statements (i)-(ii) can indeed be chosen in a way so that all required relations are satisfied. To satisfy these relations, we pick

$$
\alpha = \frac{\beta+1}{2}, \qquad \gamma = \frac{2\beta+1}{4}, \qquad \nu = \beta(1-\beta), \qquad \mu = \frac{1-\beta}{2}. \tag{49}
$$

For this choice of parameters, the inequalities (25), (29), and (31) are satisfied. Furthermore, (39) and (42) are also satisfied if

$$
\delta_0 < \frac{1-\beta}{64 B_0}, \tag{50}
$$

and (44) is satisfied if

$$\delta_0 < \frac{\lambda\left(3\beta - 2\beta^2 - 1\right)}{B_0\left(9\beta - 6\beta^2 + 41\right)}. \tag{51}$$

Finally, condition (48) requires that

$$\delta_0 < \frac{\lambda\left(\beta^2 - 2\beta + 1\right)}{3B_0\left(-\beta^2 + 2\beta + 3\right)}. \tag{52}$$

Therefore, $\delta_0 > 0$ must be smaller than the minimum of the right-hand-side of the inequalities in (20), (39), (50),(51) and (52). This completes the proof of (i)-(ii).

The proof of statement (iii) is very similar to that of (ii), so we only outline the necessary steps. From the normal form (6) we see that the derivatives of the components of the solution $x(t)$ with respect to $\varepsilon \equiv \epsilon^\mu$ satisfy the equations

$$\begin{aligned}
\tfrac{d}{dt}\left(D_\varepsilon w_1\right) =& \left[-\lambda + \langle Y_1, w\rangle + \langle Y_2, \zeta\rangle + \sqrt{\epsilon}Y_3\right] D_\varepsilon w_1 + \Big[\langle DY_1 D_\varepsilon x, w\rangle \\
&+ \langle Y_1, D_\varepsilon w\rangle + \langle DY_2 D_\varepsilon x, \zeta\rangle + \langle Y_2, D_\varepsilon \zeta\rangle + \sqrt{\epsilon} DY_3 D_\varepsilon x + \frac{\epsilon^{\frac{1-2\mu}{2\mu}}}{2\mu} Y_3\Big] w_1, \\
\tfrac{d}{dt}\left(D_\varepsilon w_2\right) =& \left[\ \lambda + \langle Y_4, w\rangle + \langle Y_5, \zeta\rangle + \sqrt{\epsilon}Y_6\right] D_\varepsilon w_2 + \Big[\langle DY_4 D_\varepsilon x, w\rangle + \langle Y_4, D_\varepsilon w\rangle \\
&+ \langle DY_5 D_\varepsilon x, \zeta\rangle + \langle Y_5, D_\varepsilon \zeta\rangle + \sqrt{\epsilon} DY_6 D_\varepsilon x + \frac{\epsilon^{\frac{1-2\mu}{2\mu}}}{2\mu} Y_6\Big] w_2, \\
\tfrac{d}{dt}\left(D_\varepsilon \zeta\right) \ =& \ A D_\varepsilon \zeta + \left(DZ_1 D_\varepsilon x\, \zeta\right)\zeta + (Z_1\zeta) D_\varepsilon \zeta + \left(Z_1 D_\varepsilon \zeta\right)\zeta + \sqrt{\epsilon}\,\langle DZ_2 D_\varepsilon x,\ \zeta\rangle \\
&+ \sqrt{\epsilon}\,\langle Z_2,\ D_\varepsilon\, \zeta\rangle + \frac{\epsilon^{\frac{1-2\mu}{2\mu}}}{2\mu}\,\langle Z_2,\ \zeta\rangle + DZ_3 D_\varepsilon x w_1 w_2 + Z_3 D_\varepsilon(w_1 w_2), \\
\tfrac{d}{dt}\left(D_\varepsilon \rho\right) \ =& \ \sqrt{\epsilon} DE_3 D_\varepsilon x + \frac{\epsilon^{\frac{1-2\mu}{2\mu}}}{2\mu} E_3, \\
\tfrac{d}{dt}\left(D_\varepsilon \psi\right) \ =& \ \left(DF_1 D_\varepsilon x\, \zeta\right)\zeta + \left(F_1 D_\varepsilon \zeta\right)\zeta + \left(F_1 \zeta\right) D_\varepsilon \zeta + \sqrt{\epsilon}\,\langle DF_2, D_\varepsilon x\rangle + \frac{\epsilon^{\frac{1-2\mu}{2\mu}}}{2\mu} F_2 \\
&+ \langle DF_3, D_\varepsilon x\rangle\, w_1 w_2 + F_3 D_\varepsilon(w_1 w_2).
\end{aligned} \tag{53}$$

As in the proof of statement (i), we can assume that for $t \in [0, T_0)$ and $\epsilon > 0$ sufficiently small,

$$\left|D_\varepsilon \zeta(t)\right| \le \bar{B}'_\zeta \epsilon^\gamma, \quad \left|D_\varepsilon \rho(t)\right| \le \bar{K}'_\rho \epsilon^\mu, \quad \left|D_\varepsilon \psi(t)\right| \le \bar{K}'_\psi \epsilon^\mu, \tag{54}$$

$$\begin{aligned}
\left|D_\varepsilon [w_1(t) w_2(t)]\right| \le \bar{K}'_0 \epsilon^\beta,\ \ \left|D_\varepsilon w_1(t)\right| \le \bar{K}'_w \epsilon^\beta,\ \left|D_\varepsilon w_2(t)\right| \\
\le \bar{K}'_w \epsilon^{-\nu},\ \left\|D_\varepsilon x(t)\right\| \le 2\bar{K}'_w \epsilon^{-\nu}.
\end{aligned} \tag{55}$$

From (53), in the same way as in (29), (32), (38), (41), and (43), we obtain that the estimates in (54)-(55) continue to hold for $t \le T^*$. (To see this one only has to note that $\epsilon^{\frac{1}{2}-\mu+\beta} < \epsilon^\beta < \epsilon^\gamma$, and $\epsilon^{\frac{1}{2}-\mu} < \epsilon^\nu$.) Calculations similar to those leading to (46) now give

$$\begin{aligned}
D_\varepsilon w_1^* &= -\frac{\dot{w}_1(T^*)}{\dot{w}_2(T^*)} D_\varepsilon w_2(T^*) + D_\varepsilon w_1(T^*), \\
D_\varepsilon \zeta^* &= -\frac{\dot{\zeta}(T^*)}{\dot{w}_2(T^*)} D_\varepsilon w_2(T^*) + D_\varepsilon \zeta(T^*), \\
D_\varepsilon \rho^* &= -\frac{\dot{\rho}(T^*)}{\dot{w}_2(T^*)} D_\varepsilon w_2(T^*) + D_\varepsilon \rho(T^*), \\
D_\varepsilon \psi^* &= -\frac{\dot{\psi}(T^*)}{\dot{w}_2(T^*)} D_\varepsilon w_2(T^*) + D_\varepsilon \psi(T^*).
\end{aligned} \tag{56}$$

Then, just as in (47), we obtain from (54)-(56) the estimates listed in statement (iii) of the lemma. □

It is important to note that in the proof of the above lemma we made no use of the fact that our original system (1) is $\mathcal{O}(\epsilon)$-close to a Hamiltonian system. As it will turn out later, this fact enables us to refine some of the estimates in Lemma 4.2 for a special class of initial conditions.

5. Local and Global Maps

Lemma 4.2 shows that the "local map" $x_0 \mapsto x^*(x_0)$, as well as its partial derivatives remain bounded in the limit $\epsilon \to 0$. This enables us to extend the local map to the limit $\epsilon = 0$ so that the extension is differentiable in ϵ^μ at $\epsilon = 0$. To make this idea more precise, for $\epsilon \geq 0$ and fixed $\delta_0 > 0$ we introduce the set

$$\mathcal{L}_\epsilon = \{(w, \zeta, \rho, \psi) \in \partial_1 U_0 \cap W^u(\Pi) \mid |w_1| = \delta_0, \\ \frac{c_2 \epsilon}{\delta_0} \leq |w_2| \leq \frac{c_3 \epsilon}{\delta_0},\ |\zeta| \leq c_1 \epsilon^\beta,\ |\rho| \leq c_4\}. \tag{57}$$

$\mathcal{L}_\epsilon$ is a subset of the unstable manifold of Π whose points satisfy the entry conditions in (9). In general, $\mathcal{L}_\epsilon$ is the disjoint union of two-dimensional manifolds, and these manifolds collapse to the single two-dimensional manifold

$$\mathcal{L}_0 = \partial_1 U_0 \cap W^s_{\text{loc}}(\Pi)$$

for $\epsilon = 0$. For $\epsilon > 0$, we define the *local map* $L_\epsilon \colon \mathcal{L}_\epsilon \to \partial_1 U_0$ as

$$L_\epsilon(\delta_0, w_{20}, \zeta_0, \rho_0, \psi_0) = (w_1^*, \delta_0, \zeta^*, \rho^*, \psi^*) \tag{58}$$

with the coordinates defined as in Lemma 4.2. By the smoothness of the flow with respect to t, for $\epsilon > 0$ the map L_ϵ is of class C^r. For $\epsilon \geq 0$ we now define the map $L_0 \colon \mathcal{L}_\epsilon \to \partial_1 U_0$ as

$$L_0(\delta_0, w_{20}, \zeta_0, \rho_0, \psi_0) = (0, \delta_0, \zeta_0, \rho_0, \psi_0).$$

Note that this map simply projects any point to the local unstable manifold $W^s_{\text{loc}}(\mathcal{M}_\epsilon)$ and pushes the projection along an unstable fiber to the intersection of the fiber with $\partial_1 U_0$. Clearly, L_0 is a smooth map. Furthermore, a consequence of Lemma 4.2 is the following result.

Proposition 5.1. *For $\epsilon > 0$ small enough and for $1/2 < \beta < 1$ in the entry conditions (9), there exists $0 < \mu < 1/2$ such that the local map can be written as*

$$L_\epsilon(x_0) = L_0(x_0) + \epsilon^\mu L_1(x_0, \epsilon^\mu),$$

where L_1 is C^1 in its arguments and $L_1(x_0; 0) = 0$.

The statement of this proposition follows directly from Lemma 4.2, since the solution-dependent constants K and μ appearing in the statement of the lemma can be chosen uniformly for $x_0 \in \mathcal{L}_\epsilon$ by the compactness of the closure of Π.

Remark 5.1. It is also easy to see from (58) that the formal extension L_0 of the local map is C^r in δ_0 in a neighborhood of $\delta_0 = 0$. In this limit, the domain of L_0 becomes $\mathcal{L}_0 = \Pi$.

We now have a good approximation for the local map L_ϵ when restricted to initial conditions in the unstable manifold of the plane Π. We also want to follow initial conditions as they leave one of the faces $\{|w_2| = \delta_0\}$ of the box U_0 and return to some other face with $|w_1| = \delta_0$. Such a global excursion starts from the set

$$\mathcal{G}_\epsilon = \{(w, \zeta, \rho, \psi) \in \partial_1 U_0 \cap W^u(\Pi) \mid |w_2| = \delta_0,\ |w_1| < K\epsilon^\beta,\ |\zeta| \le K\epsilon^\beta\}, \tag{59}$$

and is described by the *global map* $G_\epsilon\colon \mathcal{G}_\epsilon \to \partial_1 U_0$ defined as

$$G_\epsilon(w_1^*, \delta_0, \zeta^*, \rho^*, \psi^*) = (\delta_0, w_{20}, \zeta_0, \rho_0, \psi_0). \tag{60}$$

The constant $K > 0$ appearing in the definition of $\mathcal{G}_\epsilon$ is the same as in statement (i) of Lemma 4.2. An approximation for the global map is given in the following lemma.

Lemma 5.2. *For $\epsilon \ge 0$ and for all sufficiently small $\delta_0 \ge 0$, the global map can be written as*

$$G_\epsilon(x^*) = x^* + \Delta x + \delta_0 G_1(x^*, \delta_0) + \sqrt{\epsilon} G_2(x^*, \epsilon),$$

where G_j are C^1 in their arguments, and the vector Δx is defined in (4).

Proof. We first observe that the map $G_0\colon \mathcal{G}_0 \to \Pi$ remains well-defined in the limit $\delta_0 = 0$ with domain $\mathcal{G}_0 = \Pi$. The map G_0 simply relates the α-limit points of unperturbed heteroclinic orbits in $W^u(\mathcal{C}) \equiv W^s(\mathcal{C})$ to their ω-limit points. Therefore, for $\delta_0 = 0$ we obtain $G_0(x^*) = x^* + \Delta x$ from assumption (H6). For nonzero $\delta_0 > 0$, G_0 maps the first intersections of solutions in the homoclinic manifolds $W_0^\pm(\mathcal{C})$ with ∂U_0 to their second intersections with ∂U_0. Since these solutions locally coincide with unperturbed fibers in $W^{s,u}_{\text{loc}}(\mathcal{C})$, and fibers depend smoothly on their basepoints, we obtain that $G_0(x^*) = x^* + \Delta x + \delta_0 G_1(x^*, \delta_0)$. Now by assumption (H2), for $x^* \in \mathcal{G}_\epsilon$, the global map $G_\epsilon(x^*)$ is smooth in the initial condition x^* and the parameter $\sqrt{\epsilon}$. Initial conditions in the domain of G_0 are at most $\mathcal{O}(\epsilon^\beta)$ (with $\beta > 1/2$) away from $\mathcal{G}_\epsilon$, and the magnitude to the perturbation in the Fenichel normal (6) is of order $\mathcal{O}(\sqrt{\epsilon})$. This proves the statement of the lemma. □

6. Energy Estimates

In this section we shall study how the conservation of the Hamiltonian $H = H_0 + \epsilon H_1$ is violated on solutions due to the presence of general dissipative terms in Eq. (1). The reason for this study is that we shall use the "energy" H together with the normal form variables (w_2, ζ, ρ, ψ) as coordinates to identify solutions entering the set U_0 through its face $w_1 = \delta_0$. Similarly, we shall use the coordinates $(H, w_1, \zeta, \rho, \psi)$ to label solutions that leave U_0 through its face $w_2 = \delta_0$.

We start with some preliminary estimates which will be needed in our main energy estimate.

Lemma 6.1. *Let us fix a constant* $\frac{1}{2} < \beta < 1$ *and assume that for* $0 < \epsilon < \epsilon_0$ *and* $\delta_0 < \delta_1$*, the estimates (9) hold for a solution* $x(t)$ *which enters the set* U_0 *at* $t = 0$ *and leaves it at* $t = T^*$*. Then there exist constants* $L > 0$ *and* $\delta_0^* > 0$*, and for any* $\delta_0 < \delta_0^*$ *there exists* $\epsilon_0^* > 0$ *such that for all* $0 < \epsilon < \epsilon_0^*$ *we have*

$$\int_0^{T^*} |\zeta(t)|\ dt < L\sqrt{\epsilon},\ \int_0^{T^*} |w_1(t)|\ dt < L\delta_0,\ \int_0^{T^*} |w_2(t)|\ dt < L\delta_0,\ \int_0^{T^*} |\rho(t)|\ dt < L\epsilon^{\mu}, \tag{61}$$

where $\mu = (1-\beta)/2$ *(see (49)).*

Proof. The proof of this lemma is elementary, as it follows directly from the normal form (6) and the entry conditions (9). The reader may consult Haller [21] for details. □

We now formulate our main energy estimate for solutions that lie in the unstable manifold of the invariant manifold Π and make repeated passages near Π.

Lemma 6.2. *Suppose that* $x(t)$ *is a solution of the normal form (6), which lies in the unstable manifold of the invariant manifold* Π*. Let* q_0 *be the first intersection of* $x(t)$ *with the surface* $\partial_1 U_0$ *and let* $b_\epsilon = b_0 + (0, \sqrt{\epsilon}\eta) \in \Pi$ *with* $b_0 \in (\phi_0, 0) \in \mathcal{C}$ *be the basepoint of the unstable fiber* $f^u(b_\epsilon)$ *which contains the point* q_0*. Let* $x^i(t)$*,* $i = 1, \ldots, N$ *be a chain of unperturbed heteroclinic orbits for the system (1) (see Fig. 1) such that*

$$\lim_{t\to-\infty} x^1(t) = b_0, \quad \lim_{t\to+\infty} x^{i-1}(t) = \lim_{t\to-\infty} x^i(t), \qquad i = 2, \ldots, N.$$

Suppose that the solution returns to $\partial_1 U_0$ N *times to intersect it in the points* $p_1, \ldots, p_N$*, and to leave it again at the points* $q_1, \ldots, q_{N-1}$*. Assume further that, for some constants* $\frac{1}{2} < \beta < 1$*,* $0 < \epsilon < \epsilon_0$*, and* $\delta_0 < \delta_1$*, the entry conditions (9) hold for the solution* $x(t)$ *at each entry point* p_k*. (For* $N = 1$*,* $c_2 = 0$ *is allowed in (9).)*

Then, for $\delta_0, \epsilon > 0$ *sufficiently small, we have*

$$H(p_N) = H_0|\mathcal{C} + \epsilon\left[\mathcal{H}(b_0) + \sum_{i=1}^{N}\int_{-\infty}^{\infty}\langle DH_0, g\rangle_{x^i(t)}\ dt + \mathcal{O}(\delta_0, \epsilon^{\mu})\right],$$

where $0 < \mu < \frac{1}{2}$ *, and the "slow" Hamiltonian* $\mathcal{H}$ *is the first order term in the expansion of* $(H_0 + \epsilon H_1)|\Pi$ *near the torus* $\mathcal{C}$*, i.e.,*

$$\mathcal{H} = \frac{1}{2}\left\langle \eta, D_I^2 H_0(\Pi)|_{\mathcal{C}}\eta\right\rangle + H_1|\mathcal{C}. \tag{62}$$

Proof. We start by writing $H(p_N)$ in the form

$$H(p_N) = H(b_\epsilon) + [H(q_0) - H(b_\epsilon)] + \sum_{i=1}^{N-1} H(q_i) - H(p_i) + \sum_{i=1}^{N} H(p_i) - H(q_{i-1}). \tag{63}$$

We shall estimate the four main terms of this expression separately.

To estimate the first term, we note that

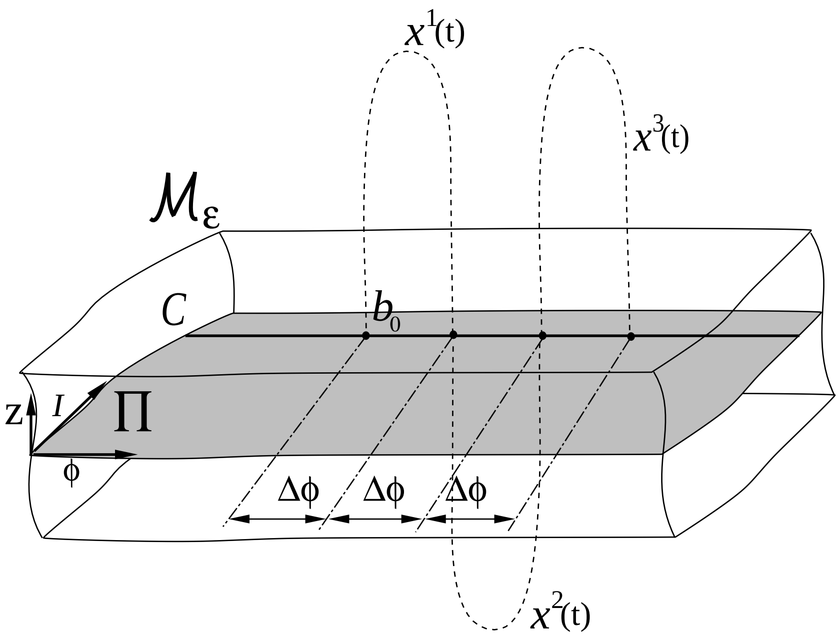

Fig. 1. The chain of heteroclinic orbits $x^i(t)$

$$H(b_\epsilon) = (H_0 + \epsilon H_1)\,|_{b_\epsilon} = H_0|\mathcal{C} + \epsilon\mathcal{H}(b_0) + \mathcal{O}(\epsilon^{3/2}), \tag{64}$$

where we used the fact that $DH_0|\mathcal{C} = 0$ since the torus $\mathcal{C}$ is filled with equilibria for $\epsilon = 0$.

To estimate the second term in (63), we consider the “Hamiltonian” unstable fiber $f^u_{g=0}(b_\epsilon)$, which intersects the surface $\partial_1 U_0$ at a point $\bar{q}_0$. Then, we have $H(\bar{q}_0) = H(b_\epsilon)$, and the mean value theorem implies that

$$|H(q_0) - H(b_\epsilon)| = |H(q_0) - H(\bar{q}_0)| < |DH(\hat{q}) \cdot (q_0 - \bar{q}_0)|\,, \tag{65}$$

where the point $\hat{q}$ lies on the line connecting q_0 and $\bar{q}_0$. Since the unstable fibers are of class C^r in the parameter ϵ, we have

$$|q_0 - \bar{q}_0| < K_1\epsilon,$$

for some integer K_1. Furthermore, the gradient of H at the point $\hat{q}$ satisfies the estimate

$$|DH(\hat{q})| < K_2\delta_0.$$

Therefore, the inequality in (65) can be rewritten as

$$|H(q_0) - H(b_\epsilon)| < K_1K_2\delta_0\epsilon. \tag{66}$$

To estimate the third term in (63), we note that

$$\sum_{i=1}^{N-1} H(q_i) - H(p_i) = \sum_{i=1}^{N-1}\int_0^{T_i^*} \dot{H}(x(t))\,dt = \sum_{i=1}^{N-1}\int_0^{T_i^*} \left[DH\cdot\left(\omega^\sharp(DH)+\epsilon g\right)\right]_{x(t)}\,dt$$
$$= \epsilon\sum_{i=1}^{N-1}\int_0^{T_i^*} \langle DH_0, g\,\rangle_{x(t)}\;dt + \mathcal{O}(\epsilon^2\log\frac{1}{\epsilon}), \tag{67}$$

where we used the fact that, by definition, $DH \cdot \omega^\sharp(DH) = \omega\left(\omega^\sharp(DH), \omega^\sharp(DH)\right) = 0$. In (67) T_i^* denotes the time of flight for the solution $x(t)$ from the point p_i to q_i, and hence obeys the estimate (10). (Here $\nu > 0$ is the constant defined in (49) and ϵ is sufficiently small.) We shall now estimate the three terms in the integrand on the right-hand-side of (67).

Noting that $DH_0|\mathcal{C} = 0$, we obtain that if (w, ζ, ρ, ψ) are the coordinates of a point $p \in S_0$, then

$$DH_0(p) = A_1(w,\zeta,\rho,\psi)w_1 + A_2(w,\zeta,\rho,\psi)w_2 + A_3(w,\zeta,\rho,\psi)\zeta + A_4(w,\zeta,\rho,\psi)\rho \quad (68)$$

for appropriate C^{r-1} functions A_i. Using Lemma 6.1 together with (68), we obtain

$$\sum_{i=1}^{N-1} \int_0^{T_i^*} \langle DH_0, g\,\rangle_{x(t)}\, dt = \mathcal{O}(\delta_0) + \mathcal{O}(\epsilon^\mu). \quad (69)$$

But this last equation and the energy expression (67) shows that

$$\sum_{i=1}^{N-1} H(q_i) - H(p_i) = \mathcal{O}(\epsilon\delta_{0,}\epsilon^{1+\mu}), \quad (70)$$

where we used the relation (31).

To complete the proof of the lemma, it remains to estimate the last sum in the expression (63). Standard “finite-time-of-flight” Gronwall estimates imply that the perturbed solutions remain close to the chain of unperturbed solutions $\left\{x^i(t)\right\}$ outside the fixed neighborhood U_0 of the manifold $\mathcal{M}_\epsilon$. Combining this with the fact that the size of U_0 is of order $\mathcal{O}(\delta_0)$, we can compute the change in energy between the points q_{i-1} and p_i in the same way as in the first line of Eq. (67). We then obtain

$$\sum_{i=1}^{N} H(p_i) - H(q_{i-1}) = \epsilon \sum_{i=1}^{N} \int_{-\infty}^{\infty} \langle DH_0, g\rangle\,|_{x^i(t)}\, dt + \mathcal{O}(\epsilon\delta_0). \quad (71)$$

But (63), (64), (70), and (71) together prove the statement of the lemma. □

In the following lemma we estimate the energy of a point $s_N \in W^s_{\rm loc}(\mathcal{M}_\epsilon) \cap \partial_1 U_0$ which has the same (z, η, ϕ) coordinates as the point p_N on the incoming solution $x(t)$. We will use this estimate to compute the energy difference between the point p_N and its projection on the unstable manifold of $\mathcal{M}_\epsilon$.

Lemma 6.3. *Suppose that $x(t)$ is a solution of the normal form and let the points $p_1, \ldots, p_N$ and $q_0, \ldots, q_{N-1}$ be defined as in Lemma 6.2. Suppose that the assumptions of that lemma hold and $c_\epsilon \in \mathcal{M}_\epsilon$ is the basepoint of a stable fiber $f^s(c_\epsilon)$ such that for the point $s_N = f^s(c_\epsilon) \cap \partial_1 U_0$,*

$$(\zeta_{p_N}, \rho_{p_N}, \psi_{p_N}) = (\zeta_{s_N}, \rho_{s_N}, \psi_{s_N}). \quad (72)$$

Then, for the energy of the point s_N, we have the expression

$$H(s_N) = H_0|\mathcal{C} + \epsilon\mathcal{H}(b_0 + N\Delta\phi) + \mathcal{O}(\epsilon\delta_0, \epsilon^{1+\beta/2}), \quad (73)$$

where the phase shift vector $\Delta\phi$ is defined in (4) and the slow Hamiltonian $\mathcal{H}$ is defined in (62).

Proof. Since the entry estimates (9) are assumed to hold for the incoming solution $x(t)$, Eq. (72) implies that the stable fiber $f^s(c_\epsilon)$ containing s_N is locally $\mathcal{O}(\epsilon^\beta)$-close to another stable fiber with basepoint on the invariant manifold Π. By the smoothness of fibers with respect to the parameter ϵ, this implies that the basepoint c_ϵ is $\mathcal{O}(\epsilon^\beta)$ close to Π, i.e.,

$$|z_{c_\epsilon}| < K_7\epsilon^\beta. \tag{74}$$

Now s_N lies at a distance of order $\mathcal{O}(\delta_0)$ from the invariant manifold $\mathcal{M}_\epsilon$, so by the smoothness of individual stable fibers we have

$$(\eta_{c_\epsilon}, \phi_{c_\epsilon}) = (\eta_{s_N}, \phi_{s_N}) + \mathcal{O}(\delta_0). \tag{75}$$

We now relate the energy of the basepoint c_ϵ to the energy of the point s_N. Let the point s^h be the intersection of the "Hamiltonian" fiber $f^s_{g=0}(c_\epsilon)$ with the surface $\partial_1 U_0$. Then, applying the mean value inequality with some point s_* lying on the line segment connecting s_N and s^h, we can write

$$\begin{aligned}|H(s_N) - H(c_\epsilon)| &= \left|H(s_N) - H(s^h)\right| < |DH|_{s_*}\left|s_N - s^h\right| \\ &< |DH|_{s_*} K_8\epsilon < K_8K_9\delta_0\epsilon,\end{aligned}$$

which yields

$$H(s_N) = H(c_\epsilon) + \mathcal{O}(\delta_0\epsilon). \tag{76}$$

Hence, to find an approximation for the energy of the point s_N, we have to compute the energy of the fiber basepoint c_ϵ. For this purpose, we have to find the restriction $\mathcal{H}_\epsilon$ of the Hamiltonian H to the manifold $\mathcal{M}_\epsilon$.

In the original x coordinate, the manifold $\mathcal{M}_\epsilon$ is given by $x = f_0(x) + \epsilon f_1(x, \epsilon)$. A standard Taylor expansion on $\mathcal{M}_\epsilon$ shows that

$$\begin{aligned}H_0|\mathcal{M}_\epsilon &= H_0|\mathcal{M}_0 + \epsilon DH_0|\mathcal{M}_0 \cdot f_1 + \mathcal{O}(\epsilon^2) \\ &= H_0|\mathcal{C} + \epsilon\langle\eta, D_I^2 H_0(\Pi)|_{\mathcal{C}}\eta\rangle + \mathcal{O}(|z|^2, \epsilon|z|, \epsilon^{\frac{3}{2}}), \\ H_1|\mathcal{M}_\epsilon &= H_1|\mathcal{M}_0 + \mathcal{O}(\epsilon) \\ &= H_1|\mathcal{C} + \mathcal{O}(|z|, \sqrt{\epsilon}).\end{aligned}$$

As a result, we have

$$\mathcal{H}_\epsilon = H|\mathcal{M}_\epsilon = H_0|\mathcal{C} + \epsilon\mathcal{H} + \mathcal{O}(|z|^2, \epsilon|z|, \epsilon^{\frac{3}{2}}) \tag{77}$$

with the slow Hamiltonian $\mathcal{H}$ defined in (62).

Since the solution $x(t)$ travels for an $\mathcal{O}(1)$ amount of time near a chain of unperturbed trajectories described in Lemma 6.2, we know that the point q_0 is $\mathcal{O}(\sqrt{\epsilon})$-close to the unperturbed solution $x^1(t)$, and the point p_N is $\mathcal{O}(\sqrt{\epsilon}^\beta)$-close to the unperturbed solution $x^N(t)$. Since $x^N(t)$ locally coincides with an unperturbed stable fiber, the smoothness of fibers implies that the basepoint c_ϵ of the fiber containing s_N is $\mathcal{O}(\sqrt{\epsilon}^\beta)$-close to the unperturbed fiber basepoint $\lim_{t\to\infty} x^N(t)$. As a result, we obtain

$$c_\epsilon = b_0 + N\Delta\phi + \mathcal{O}(\sqrt{\epsilon}^\beta),$$

where $\Delta\phi$ is defined in (4). But this last equation together with (74), (76), and (77) yields the statement of the lemma. □

7. The Existence of Multi-Pulse Homoclinic Orbits

In this section we establish a criterion for the existence of multi-pulse homoclinic or heteroclinic orbits that are doubly asymptotic to the invariant manifold $\mathcal{M}_\epsilon$. These orbits are contained in the unstable manifold of the invariant manifold Π, and in some cases they also lie in the stable manifold of Π.

We first give an easy improvement of the results listed in Lemma 4.2 on the coordinates of the solution $x(t)$ upon its exit from the set U_0. This improvement makes use of the energy estimates in Lemma 6.2. The result is that the change in the coordinates w_1 and ζ during local passages near $\mathcal{M}_\epsilon$ is of the order $\mathcal{O}(\epsilon)$ if the solution $x(t)$ satisfies the entry conditions (9) and lies in the unstable manifold of the manifold Π. This is due to the Hamiltonian nature of the unperturbed problem, which was not used in the derivation of the general normal form (6).

Lemma 7.1. *Let us fix a constant* $\frac{1}{2} < \beta < 1$ *and assume that a solution* $x(t)$ *of the normal form (6) enters the set* U_0 *at* $t = 0$ *and leaves it at* $t = T^*$*. Assume further that* $x(t)$ *is contained in the manifold* $W^u(\Pi)$ *and satisfies the entry conditions (9). Let us introduce the notation* $a = (w_{20}, \zeta_0, \rho_0, \psi_0)$ *, and let* $x_0 = (\delta_{0,}a)$ *and* $x^* = x(T^*) = (w_1^*, \delta_0, \zeta^*, \rho^*, \psi^*)$ *define the coordinates of the solution at entry and departure, respectively.*

Then there exist constants $K > 0$ *,* $0 < \mu < \frac{1}{2}$*, and* $\delta_0^* > 0$*, and for any* $\delta_0 < \delta_0^*$ *there exists* $\epsilon_0^* > 0$ *such that for all* $0 < \epsilon < \epsilon_0^*$ *the following estimates hold:*

(i)

$$|w_1^*| < K\epsilon,\ |\zeta^*| < K\epsilon,\ |\rho^* - \rho_0| < K\sqrt{\epsilon}^{\beta},\ |\psi^* - \psi_0| < K\sqrt{\epsilon}^{\beta}.$$

(ii)

$$\begin{array}{ll} |D_a\, w_1^*| < K\epsilon^{\beta}, & |D_a\, \zeta^* - (0,1,0,0)| < K\epsilon^{\mu}, \\ |D_a\, \rho^* - (0,0,1,0)| < K\epsilon^{\mu}, & |D_a \psi^* - (0,0,0,1)| < K\epsilon^{\mu}. \end{array}$$

(iii)

$$|D_{\epsilon^\mu}\, w_1^*| < K\epsilon^{\beta}, \quad |D_{\epsilon^\mu} \zeta^*| < K\epsilon^{\mu}, \quad |D_{\epsilon^\mu}\, \rho^*| < K\epsilon^{\mu}, \quad |D_{\epsilon^\mu} \psi^*| < K\epsilon^{\mu}.$$

Proof. Consider the point $q^* \in W^u_{\text{loc}}(\Pi)$ for which $w_{1q^*} = 0$, $\zeta_{q^*} = 0$, and $(\rho_{q^*}, \psi_{q^*}) = (\rho_{x^*}, \psi_{x^*})$ hold. By (i) of Lemma 4.2, the points q^* and x^* are $\mathcal{O}(\epsilon^\beta)$ close. To determine the energy of the point q^*, we consider the unstable fiber $f^u(b^*)$ which contains q^*. For zero dissipation ($g \equiv 0$), the energy of the basepoint b^* of the fiber $f^u_{g=0}(b^*)$ can be written in the form $H(b^*) = H_0|\mathcal{C} + \mathcal{O}(\epsilon)$,where we used (77). Since the energy is constant on fibers for $g \equiv 0$, we immediately obtain

$$H(q^*) = H_0|\mathcal{C} + \mathcal{O}(\epsilon). \tag{78}$$

This equation remains valid for nonzero dissipation, since unstable fibers perturb by an $\mathcal{O}(\epsilon)$ amount when we add the dissipative terms. Also, setting $q_1 = x^*$ in Lemma 6.2, we obtain that

$$H(x^*) = H_0|\mathcal{C} + \mathcal{O}(\epsilon).$$

This last equation together with (78) and the mean value inequality gives

$$K_{10}\epsilon > |H(q^*) - H(x^*)| = \left| DH(\hat{q}) \cdot \frac{q^* - x^*}{|q^* - x^*|} \right| |q^* - x^*| > K_{11}\delta_0\, |q^* - x^*|\,, \tag{79}$$

where $\hat{q}$ is an appropriate point on the line connecting the points q^* and x^*. Here we made use of the facts that the diameter of the set U_0 is of the order $\mathcal{O}(\delta_0)$ and the perturbed flow intersects the line between q^* and x^* with $\mathcal{O}(1)$ transversality due to the geometry of the unperturbed Hamiltonian flow. We rewrite (79) in the form

$$|q^* - x^*| < \frac{K_{10}}{K_{11}\delta_0}\epsilon. \tag{80}$$

Since the transformation from the (y, z, η, ϕ) coordinates to the (w, ζ, ρ, ψ) coordinates is a diffeomorphism with norm of order $\mathcal{O}(1)$, this last expression implies that

$$|w_1^*| < K_{12}\epsilon, \tag{81}$$

since $w_{1q^*} = 0$. Furthermore, as the unstable fibers are straight lines for the local normal form (6), $\zeta_{q^*} = 0$ must hold, since the basepoint of the unstable fiber containing q^* lies in the invariant manifold Π which obeys $\zeta = 0$. As a result, (81) implies

$$|\zeta^*| < K_{13}\epsilon,$$

which, together with the estimate (81) proves the first two inequalities in statement (i) of the lemma. The remaining inequalities are just restatements of the results listed in Lemma 4.2. □

The following definition describes the types of orbits that we will be interested in finding.

Definition 7.1. *Let us consider a point $b_0 \in \mathcal{C}$ and let $j = \{j_i\}_{i=1}^N$ be a sequence of $+1$'s and -1's. An orbit x_ϵ of system (1) is called an N-*pulse homoclinic orbit with basepoint b_0 and jump sequence j, *if for some $0 < \mu < \frac{1}{2}$ and for $\epsilon > 0$ sufficiently small,*

(i) x_ϵ intersects an unstable fiber $f^u(b_\epsilon)$ with basepoint $b_\epsilon = b_0 + \mathcal{O}(\epsilon^\mu) \in \Pi$,
(ii) x_ϵ intersects a stable fiber $f^s(c_\epsilon)$ with basepoint $c_\epsilon = b_0 + N\Delta\phi + \mathcal{O}(\epsilon^\mu) \in \mathcal{M}_\epsilon$ such that $\mathrm{dist}(c_\epsilon, \Pi) = \mathcal{O}(\epsilon)$.
(iii) Outside a small fixed neighborhood of the manifold $\mathcal{M}_\epsilon$, the orbit x_ϵ is order $\mathcal{O}(\epsilon^\mu)$ close to a chain of unperturbed heteroclinic solutions $x^i(t)$, $i = 1, \dots, N$, such that

$$\lim_{t\to-\infty} x^1(t) = b_0, \qquad \lim_{t\to+\infty} x^{i-1}(t) = \lim_{t\to-\infty} x^i(t),\ i = 2, \dots, N.$$

Furthermore, for $k = 1, \dots, N$ and for all $t \in \mathbb{R}$ we have

$$x^k(t) \in \begin{cases} W_0^+(\mathcal{C}) \text{ if } j_k = +1, \\ W_0^-(\mathcal{C}) \text{ if } j_k = -1. \end{cases}$$

To illustrate the above definition, we show a three-pulse homoclinic orbit schematically in Fig. 2.

To find N-pulse orbits of the type described in Definition 7.1, it is clearly enough to find conditions under which the points $p_N \in W^u(\Pi)$ and $s_N \in W^s_{\mathrm{loc}}(\mathcal{M}_\epsilon)$ coincide. By construction, these points have the same w_1, ζ, ρ, and ψ coordinates, so they coincide if their w_2 coordinates are equal, i.e., the w_2 coordinate of p_N is zero. However, instead of following the evolution of the w_2 coordinate along solutions, we will follow the change of "energy" H along solutions. The following lemma shows that this is sufficient, since the w_2 coordinate of p_N can uniquely be determined as a function of the other coordinates and $H(p_N)$. This result will enable us to detect N-pulse orbits by solving the equation $H(p_N) - H(s_N) = 0$.

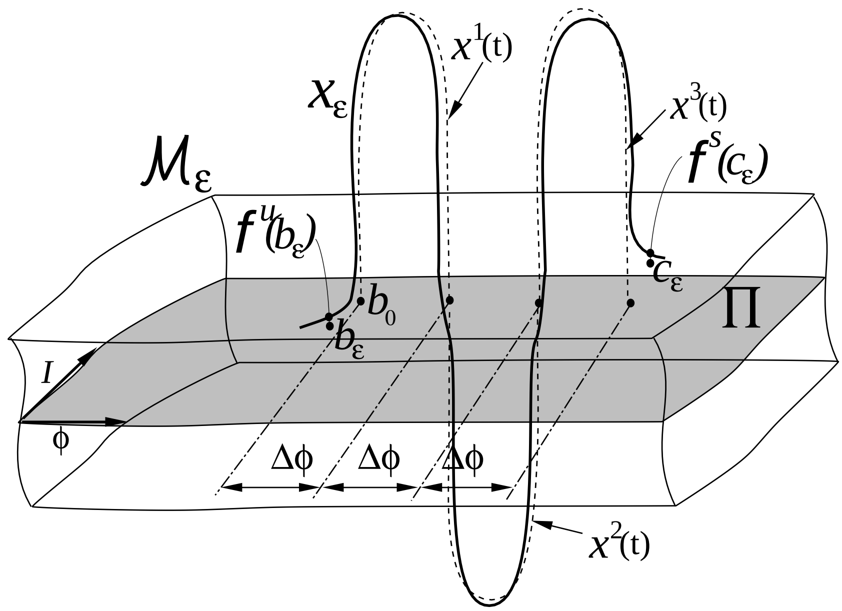

Fig. 2. 3-pulse homoclinic orbit to the manifold $\mathcal{M}_\epsilon$ with jump sequence $j = \{+1, -1, +1\}$ and with base-point b_ϵ

Lemma 7.2. *Suppose that the conditions of Lemma 6.2 are satisfied. Then for $\epsilon > 0$ small enough there exists a C^1 function $f_\epsilon : \mathcal{P} \mapsto \mathbb{R}$, such that for any $l = 1, \ldots N$,*

$$w_{2p_l} = f_\epsilon \left(\zeta_{p_l}, \rho_{p_l}, \psi_{p_l}, H(p_l) \right) .$$

Proof. We start by noting that, in terms of the original x coordinate used in Eq. (1), the surface $\{w_1 = \delta_0\}$ is given in the form $x = s_\epsilon(w_2, \zeta, \rho, \psi)$, where s_ϵ is a C^r embedding into the space $\mathcal{P}$. Then the intersection of the energy surface $\{H(x) = h\}$ with $\{w_1 = \delta_0\}$ satisfies the equation

$$H(s_\epsilon(w_2, \zeta, \rho, \psi)) - h = 0.$$

By the implicit function theorem, on this intersection set the coordinate w_2 is a C^1 function of the rest of the coordinates and the energy h provided

$$\langle DH(s_\epsilon(w_2, \zeta, \rho, \psi)), D_{w_2} s_\epsilon(w_2, \zeta, \rho, \psi) \rangle \neq 0 \tag{82}$$

holds in all points of the intersection. We want to see if this equation holds at the point p_l. Since $p_l \to s_l$ as $\epsilon \to 0$, and p_l is contained in a compact subset of $W^u(\Pi)$, it is enough to verify that

$$|\langle DH_0(s_l), D_{w_2} s_0(w_{2s_l}, \zeta_{s_l}, \rho_{s_l}, \psi_{s_l}) \rangle| > c_l \tag{83}$$

for some constant $c_l > 0$. But the vector $D_{w_2} s_0(w_{2s_l}, \zeta_{s_l}, \rho_{s_l}, \psi_{s_l})$ lies in the tangent space of $\partial_1 U_0$, so this last inequality follows from the fact that the unperturbed flow intersects $\partial_1 U_0$ with $\mathcal{O}(1)$ transversality. Thus the statement of the lemma follows by the implicit function theorem. □

We are now in the position to prove our main result on the existence of solutions backward asymptotic to the invariant manifold Π and forward asymptotic to the manifold $\mathcal{M}_\epsilon$. The key ingredient we shall need is the *N-th order energy-difference function* $\Delta^N\mathcal{H}$. For any point $b_0 = (\eta, \phi) \in \mathcal{C}$, this function is defined as

$$\boxed{\Delta^N\mathcal{H}(\phi) = \mathcal{H}(b_0 + N\Delta\phi) - \mathcal{H}(b_0) - \sum_{i=1}^{N}\int_{-\infty}^{\infty} \langle DH_0, g\rangle\,|_{x^i(t)}\; dt,} \tag{84}$$

where $N \geq 1$ is an integer, the slow Hamiltonian $\mathcal{H}$ is defined in (62), the phase shift vector $\Delta\phi$ is defined in (4), and $x^i(t)$, $i = 1, \ldots, N$ is a chain of unperturbed heteroclinic solutions as described in Lemma 6.2 with

$$\lim_{t\to-\infty} x^1(t) = x.$$

Finally, we introduce a definition which will be used to determine the jump sequences of multi-pulse homoclinic orbits. To this end, let us consider a point p^+ on the unperturbed homoclinic manifold $W_0^+ \equiv W_0^+(\mathcal{M}_0)$. Since W_0^+ is a hypersurface in the phase space $\mathcal{P}$, it makes sense to define the vector $\mathbf{n}(p^+)$ as the unit normal to W_0^+ which points in the direction of the other unperturbed homoclinic manifold $W_0^- \equiv W_0^-(\mathcal{M}_0)$. (See Fig. 3 for a schematic picture.)

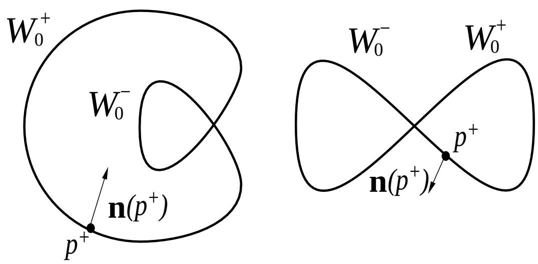

Fig. 3. The definition of the vector $\mathbf{n}(p^+)$ in two different cases

This allows us to introduce the number

$$\sigma = \text{sign}\,\left(DH_0 \cdot \mathbf{n}(p^+)\right). \tag{85}$$

Note that σ is independent of the choice of the point p^+ by the normal hyperbolicity of the unperturbed manifold $\mathcal{M}_0$. Furthermore, σ remains the same if we interchange the roles of the homoclinic manifolds W_0^+ and W_0^- in this construction. It is easy to see that $\sigma = -1$ ($\sigma = +1$) if the energy H_0 of the unperturbed solutions encircled by the homoclinic manifold is higher (lower) than the energy of those lying outside of W_0^+. This meaning of σ is preserved under small perturbations.

Definition 7.2. *For any value* $\phi_0 \in \mathbb{T}^m$, **the positive sign sequence** $\chi^+(\phi_0) = \{\chi_k^+(\phi_0)\}_{k=1}^N$ *is defined as*

$$\chi_1^+(\phi_0) = +1, \quad \chi_{k+1}^+(\phi_0) = \sigma sign\,\left(\Delta^k\mathcal{H}(\phi_0)\right)\chi_k^+(\phi_0), \qquad k = 1, \ldots, N-1.$$

The **negative sign sequence** $\chi^-(\phi_0) = \{\chi_k^-(\phi_0)\}_{k=1}^N$ *is defined as*

$$\chi^-(\phi_0) = -\chi^+(\phi_0).$$

We now formulate our main result on the existence of N-pulse homoclinic orbits for the perturbed system (1).

Theorem 7.3. *Suppose that for some positive integer N, $\phi_0 \in \mathbb{T}^m$ is a transverse zero of the function $\Delta^N \mathcal{H}$, i.e., after a possible reindexing of the angular variables ϕ we have*

$$\Delta^N \mathcal{H}(\phi_0) = 0, \quad D_{\phi_1} \Delta^N \mathcal{H}(\phi_0) \neq 0.$$

Suppose further that $\Delta^k \mathcal{H}(\phi_0) \neq 0$ holds for all integers $k = 1, \ldots, N-1$, and let $\phi = (\phi_1, \tilde{\phi})$ with $\tilde{\phi} \in \mathbb{T}^{m-1}$.

Then there exist constants $0 < \mu < \frac{1}{2}$ and $C_\eta > 0$, such that for any small enough $\epsilon > 0$, the system (2) admits two, $2m-1$–parameter families of N-pulse homoclinic orbits $x_\epsilon^\pm(\tilde{\phi}, \eta_0)$ with basepoints $b_\epsilon^\pm(\tilde{\phi}, \eta_0) \in \Pi$ such that

$$b_\epsilon^\pm(\tilde{\phi}, \eta_0) = (\phi_0 + \mathcal{O}(\epsilon^\mu), I_0 + \sqrt{\epsilon}\eta_0).$$

Here $|\eta_0| < C_\eta$ is an arbitrary localized action value. The jump sequences of the orbits are given by $\chi^\pm(\phi_0)$, respectively. Furthermore, the basepoints $b_\epsilon^\pm$ depend on $\tilde{\phi}$ and ϵ^μ in a C^1 fashion.

Proof. For $\epsilon > 0$ and $\delta_0 > 0$ sufficiently small, let us consider a solution $x(t)$ which lies in the component $W_0^{u+}(\Pi)$ of the unstable manifold of the invariant manifold Π. ($W_0^{u+}(\Pi)$ denotes the connected component of $W_0^u(\Pi)$ that perturbs from the homoclinic manifold W_0^+.) We follow $x(t)$ up to its first intersection with the surface $w_2 = \delta_0$. We denote this intersection point by q_0 and note that it lies on an unstable fiber $f^u(b_\epsilon)$ with some basepoint $b_\epsilon = (\phi_0, \sqrt{\epsilon}\eta_0) \in \Pi$ (see Fig. 4). We then follow the solution

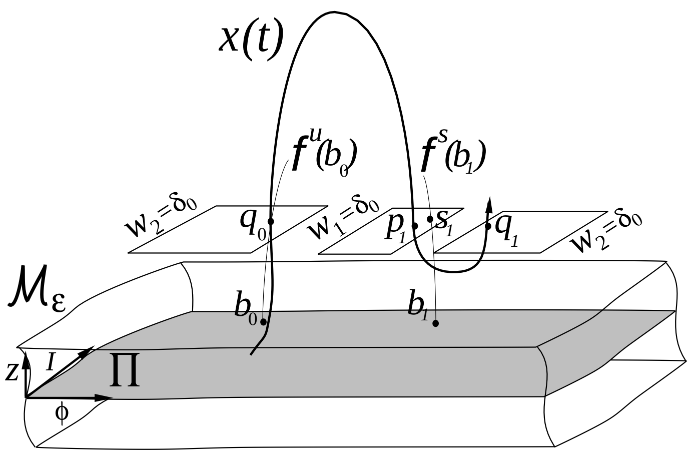

Fig. 4. The geometry of the proof of Theorem 6.2

as it leaves the neighborhood U_0 of the manifold $\mathcal{M}_\epsilon$ and, by standard Gronwall estimates, returns and intersects the subset $|w_1| = \delta_0$ of the surface $\partial_1 U_0$. We denote this second intersection point by p_1 (see Fig. 4). Since the unstable fibers are straight in the (w, ζ, ρ, ψ) coordinates, we have $|\zeta_{q_0}| = 0$ by construction. Then q_0 is clearly contained in the domain $\mathcal{G}_\epsilon$ of the global map G_ϵ (see (59) and (60)) and we can write $p_1 = G_\epsilon(q_0)$.

Since the manifold $W_{\text{loc}}^{s+}(\mathcal{M}_\epsilon)$ is a graph over the variables (w_1, ζ, ρ, ψ), there exists a unique point $s_1 \in W_{\text{loc}}^{s+}(\mathcal{M}_\epsilon) \cap \partial_1 U_0$ as defined in Lemma 6.3. In particular, we have

$$\left(\zeta_{s_1}, \rho_{s_1}, \psi_{s_1}\right) = \left(\zeta_{p_1}, \rho_{p_1}, \psi_{p_1}\right).$$

According to Lemma 7.2, $p_1 \equiv s_1$ holds if and only if

$$H(p_1) - H(s_1(p_1)) = 0, \tag{86}$$

where we view s_1 as a function of p_1. Note that the right-hand-side of Eq. (86) is C^r in the variable p_1.

By standard Gronwall estimates, the point p_1 of the solution $x(t)$ is $\mathcal{O}(\epsilon)$-close to a stable fiber $f^s(b_1)$ with basepoint $b_1 = b_\epsilon + \Delta\phi \in \Pi$ (see Fig. 7). As a result, it satisfies the entry conditions listed in (9) with $\beta = 1$ and $c_2 = 0$. Consequently, Lemma 6.2 applies with $n = 1$ and gives

$$H(p_1(b_\epsilon)) = H_0|\mathcal{C} + \epsilon\left[\mathcal{H}(b_0) + \int_{-\infty}^{\infty} \langle DH_0, g\rangle_{x^1(t)} + \mathcal{O}(\delta_0, \epsilon^\mu)\right] \tag{87}$$

for an appropriate constant $0 < \mu < \frac{1}{2}$. Furthermore, Lemma 6.3 with $n = 1$ also applies and yields

$$H(s_1(b_\epsilon)) = H_0|\mathcal{C} + \epsilon\mathcal{H}(b_0 + \Delta\phi) + \mathcal{O}(\epsilon\delta_0, \epsilon^{\frac{3}{2}}). \tag{88}$$

Since $b_\epsilon = b_0 + \mathcal{O}(\sqrt{\epsilon}) = (\phi_0, \sqrt{\epsilon}\eta_0)$, for any $\epsilon > 0$ we can use (87), (88), and the definition of $\Delta^1\mathcal{H}$ in (84) to rewrite the energy Eq. (86) as

$$\Delta^1\mathcal{H}(\phi_0) + \delta_0\mathcal{F}_1(p_1(b_\epsilon); \delta_0, \epsilon^\mu) + \epsilon^\mu\mathcal{G}_1(p_1(b_\epsilon); \delta_0, \epsilon^\mu) = 0 \tag{89}$$

with $p_1 = (0, w_{2p_1}, \zeta_{p_1}, \rho_{p_1}, \psi_{p_1}) = G_\epsilon(q_0)$. The points b_0 and p_1 are related by

$$p_1(b_0) = G_\epsilon \circ P^u_\epsilon(b_0), \tag{90}$$

where $P^u_\epsilon\colon W^{u+}_{\text{loc}}(\Pi) \cap \partial_1 U_0 \to \Pi$ is the fiber projection map that maps the intersection points of unstable fibers in $W^{u+}_{\text{loc}}(\Pi)$ with the surface $\partial_1 U_0$ to the basepoints of these fibers. By the smoothness of fibers in $W^{u+}_{\text{loc}}(\Pi)$, the function P^u_ϵ is a C^r map. By Lemma 5.2, G_ϵ is a C^1 map from $\mathcal{G}_\epsilon$ to $\mathcal{P}$. As a result, Eq. (90) shows that p_1 is a C^1 function of b_0. This in turn implies that the right-hand-side of the energy Eq. (89) is of class C^1 with respect to b_0, because the functions $\mathcal{F}_1$ and $\mathcal{G}_1$ are smooth in p_1, as we observed after formula (86), and $\Delta^1\mathcal{H}$ is a C^1 function.

Assume now that $N = 1$ holds in the statement of the theorem. Then, by the assumptions of the theorem, $b_0 = (\phi_0, \sqrt{\epsilon}\eta_0)$ with any $0 < |\eta_0| \le C_\eta$ is a solution of Eq. (89) for $\delta_0 = \epsilon = 0$. We want to apply the implicit function theorem to argue that this solution can be continued for $\epsilon, \delta_0 > 0$. Setting $\epsilon = 0$, and differentiating (89) with respect to the ϕ_1 coordinate of b_0 yields

$$\begin{aligned} D_{\phi_1}\left[\Delta^1\mathcal{H}(\phi_0) + \delta_0\mathcal{F}_1(p_1(b_0); \delta_0, 0)\right] &= D_{\phi_1}\Delta^1\mathcal{H}(\phi_0) \\ &\quad + \delta_0\left\langle D_{p_1}\mathcal{F}_1, DG_0 DP^u_0 D_{\phi_{01}}\mathcal{T}_0^{-1}\right\rangle|_{b_0}. \end{aligned} \tag{91}$$

Here, $\phi_0 = (\phi_{01}, \tilde{\phi}_0)$ and $\mathcal{T}_\epsilon$ is the normal form transformation constructed in Lemma 3.1. Now $D_{\phi_1}\Delta^1\mathcal{H}$ is a continuous function, and we have $D_{\phi_1}\Delta^1\mathcal{H}(\phi_0) \neq 0$ by assumption. Hence for sufficiently small $\delta_0 > 0$, (91) is nonzero. (This follows by recalling that the right-hand-side of (91) continuous in b_0 and the term

$$\left\langle D_{p_1}\mathcal{F}_1, DG_0 DP^u_0 D_{\phi_{01}}\mathcal{T}_0^{-1}\right\rangle|_{b_0}$$

remains bounded as $\delta_0 \to 0$ by Lemma 5.2.) Thus (89) admits a solution $\bar{\phi}_1(\tilde{\phi}, \eta_0, \delta_0) = \phi_{01} + \mathcal{O}(\delta_0)$ for $\delta_0 > 0$ small and $\epsilon = 0$. We fix δ_0 sufficiently small, substitute the solution ϕ_1 back into Eq. (89). The derivative of the left-hand-side of the resulting equation with respect to ϕ_1 is given by

$$D_{\phi_1}\Delta^1\mathcal{H}\left((\bar{\phi}_1, \tilde{\phi})\right) + \delta_0 \left\langle D_{p_1}\mathcal{F}_1, DG_\epsilon DP_\epsilon^u D_{\phi_{01}}\mathcal{T}_\epsilon^{-1}\right\rangle$$
$$+\epsilon^\mu \left\langle \nabla_{p_1}\mathcal{G}_1, DG_\epsilon DP_\epsilon^u D_{\phi_{01}}\mathcal{T}_\epsilon^{-1}\right\rangle.$$

By Lemma 5.2, this derivative is continuous at $\epsilon = 0$, and is also nonzero by assumption. Thus Eq. (89) admits a solution $\hat{\phi}_1(\tilde{\phi}, \eta_0, \delta_0, \epsilon) = \phi_{01} + \mathcal{O}(\delta_0, \epsilon^\mu)$ for $\epsilon > 0$ sufficiently small. For any fixed ϵ, the solution should not depend on δ_0, which is just an auxiliary parameter to measure the size of the neighborhood U_0 that we have worked in. Therefore, we have $d\hat{\phi}_1/d\delta_0 = 0$, implying $\hat{\phi}_1(\tilde{\phi}, \eta_0, \epsilon) = \phi_{01} + \mathcal{O}(\epsilon^\mu)$. This proves the existence of the orbit family $x_\epsilon^+(\tilde{\phi})$ for $N = 1$. The smoothness of $x_\epsilon^+(\tilde{\phi})$ with respect to ϵ^μ follows from Lemma 5.2.

Assume now that $N > 1$ in the statement of the theorem. Then, by the conditions of the theorem, we see that for ϵ and δ_0 sufficiently small the energy Eq. (89) cannot be satisfied, so the solution $x(t)$ does not intersect the local stable manifold of $\mathcal{M}_\epsilon$ upon its first return to the neighborhood U_0. Using (87), (88), and the compactness of the solid m-torus $[-C_\eta, C_\eta]^m \times \mathbb{T}^m$, we conclude the existence of positive constants $K_1^{(1)}$ and $K_2^{(1)}$such that

$$K_1^{(1)}\epsilon < |H(p_1) - H(s_1)| < K_2^{(1)}\epsilon. \tag{92}$$

Now the mean value theorem implies for any fixed $k \geq 1$,

$$\begin{aligned}|H(p_1) - H(s_1)| &= \left|\left\langle DH(p_1^*), \frac{p_1 - s_1}{|p_1 - s_1|}\right\rangle\right| |p_1 - s_1| \\ &> C_2^{(1)} |p_1 - s_1|,\end{aligned} \tag{93}$$

where p_1^* is a point on the line connecting p_1 and s_1, and the existence of $C_2^{(1)} > 0$ follows from an argument similar to that leading to estimate (83). At the same time, the mean value theorem implies that

$$|H(p_1) - H(s_1)| < C_1^{(1)} |p_1 - s_1| \tag{94}$$

for some constant $C_1^{(1)} > 0$, so it follows from (92)-(94) that

$$\frac{K_1^{(1)}\epsilon}{C_1^{(1)}} < |p_1 - s_1| < \frac{K_2^{(1)}\epsilon}{C_2^{(1)}}. \tag{95}$$

This last expression in (95) immediately shows that the coordinates $(w_{2p_1}, \zeta_{p_1}, \rho_{p_1}, \psi_{p_1})$ satisfy the entry conditions in (9) (because the normal form coordinates of the point s_1 satisfy $w_{1s_1} = \delta_0$, $w_{2s_1} = 0$, and $|\zeta_{s_1}| = \mathcal{O}(\epsilon)$). Consequently, the point p_1 is contained in the domain $\mathcal{L}_\epsilon$ of the local map L_ϵ, and we can write $q_1 = L_\epsilon(p_1)$, where q_1 is the next intersection of the solution $x(t)$ with the surface $\partial_1 U_0$.

Let p_2 denote the intersection of the solution $x(t)$ with the surface $\partial_1 U_0$ upon its second return to the neighborhood U_0. (The existence of p_2 is guaranteed by the usual Gronwall estimates for $\epsilon > 0$ small enough.) We again have a point $s_2 \in W_{\text{loc}}^s(\mathcal{M}_\epsilon) \cap \partial_1 U_0$ such that

$$\left(\zeta_{s_2}, \rho_{s_2}, \psi_{s_2}\right) = \left(\zeta_{p_2}, \rho_{p_2}, \psi_{p_2}\right).$$

Again, the solution $x(t)$ gives rise to a 2-pulse homoclinic orbit if

$$H(p_2) - H(s_2(p_2)) = 0,$$

or, alternatively,

$$\Delta^2\mathcal{H}(\phi_0) + \delta_0\mathcal{F}_2(p_2(b_\epsilon); \delta_0, \epsilon^\mu) + \epsilon^\mu\mathcal{G}_2(p_2(b_\epsilon); \delta_0, \epsilon^\mu) = 0, \tag{96}$$

where we used Lemmas 6.2 and 6.3. As in Eq. (89), the functions $\mathcal{F}_2$ and $\mathcal{G}_2$ are C^1 in their arguments. Since

$$p_2(b_\epsilon) = G_\epsilon \circ L_\epsilon \circ G_\epsilon \circ P_\epsilon^u(b_\epsilon),$$

we see that for $\epsilon \geq 0$, p_2 is a C^1 function of b_ϵ and ϵ^μ by Corollary 5.1 and Lemma 5.2. Then just as in the case of $N = 1$, the implicit function theorem applied to (96) implies the existence of the orbit family $x_\epsilon^+(\tilde{\phi}, \eta_0)$ for $N = 2$.

The proof for any $N > 2$ follows the same steps as that of the case $N = 2$. The existence of the other N-pulse homoclinic orbit family $x_\epsilon^-(\tilde{\phi}, \eta_0)$ for any $N \geq 1$ follows from the fact that an identical construction can be repeated for solutions contained in $W^{u-}(\mathit{\Pi})$. Therefore, it remains to show that the jump sequences of the two families $x_\epsilon^\pm(\tilde{\phi})$ are indeed given by the sign sequences $\chi^\pm(\phi_0)$, respectively. We sketch the argument for x_ϵ^+ only since the argument for x_ϵ^- is identical.

Consider an N-pulse homoclinic orbit x_ϵ^+. By construction it makes its first pulse in the vicinity of the unperturbed manifold $W_0^+(\mathcal{C})$, hence the first element of its jump sequence is indeed $\chi_1^+(\phi_0) = +1$. For small $\epsilon, \delta_0 > 0$, at the first re-entry point p_1 we have

$$\begin{aligned}\text{sign }(H(s_1) - H(p_1)) &= \text{sign}\,[\epsilon(\Delta^1\mathcal{H}(\phi_0 + \mathcal{O}(\delta_0, \epsilon^\mu))\\ &\quad +\delta_0\mathcal{F}_N(p_N(b_\epsilon^+); \delta_0, \epsilon^\mu)\\ &\quad +\epsilon^\mu\mathcal{G}_N(p_N(b_\epsilon^+); \delta_0, \epsilon^\mu))]\\ &= \text{sign }\left(\Delta^1\mathcal{H}(\phi_0)\right).\end{aligned} \tag{97}$$

If this quantity is positive, then at the point p_1 the solution $x(t)$ has higher energy than nearby points in the hypersurface $W_{\text{loc}}^{s+}(\mathcal{M}_\epsilon)$. Recalling the meaning of the constant σ (see (85)), we can conclude that $\sigma\,\text{sign}\left(\Delta^1\mathcal{H}(\phi_0)\right) = +1$ implies that the solution $x(t)$ stays near the homoclinic manifold $W_0^+(\mathcal{C})$, whereas $\sigma\,\text{sign}\left(\Delta^1\mathcal{H}(\phi_0)\right) = -1$ causes the solution to perform its second jump in the vicinity of the manifold $W_0^-(\mathcal{C})$. Therefore, the second element in the jump sequence of x_ϵ^+ is given by $\chi_2^+(\phi_0)$ as defined in Definition 7.2. The remaining elements of the jump sequence of x_ϵ^+ are constructed recursively in the same fashion, hence they coincide with the corresponding elements of the sign sequence $\chi^+(\phi_0)$ in Definition 7.2. This completes the proof of the theorem. □

In the following we describe two situations in which the above theorem can be applied. For simplicity, we will consider the case $m = 1$, i.e., *we assume that the manifold $\mathit{\Pi}$ is two-dimensional*, hence the center manifold $\mathcal{M}_0$ of the unperturbed system is $2n+2$ dimensional.

To find the asymptotic behavior of multi-pulse orbits, one has to have some knowledge of the dynamics on the two-dimensional manifold $\mathcal{M}_\epsilon$. A straightforward Taylor expansion shows (see, e.g., Haller and Wiggins [15]) that near the resonant circle $\mathcal{C}$ the flow on $\mathit{\Pi}$ satisfies the equations

$$\begin{aligned}\dot{\eta} &= \sqrt{\epsilon} D_\phi \mathcal{H}_g(\eta,\phi) + \mathcal{O}(\epsilon), \\ \phi &= -\sqrt{\epsilon} D_\eta \mathcal{H}_g(\eta,\phi) + \mathcal{O}(\epsilon),\end{aligned} \tag{98}$$

with

$$\begin{aligned}\mathcal{H}_g(\eta,\phi) &= \mathcal{H}(\eta,\phi) - \int_0^\phi g_I|_{\mathcal{C}}(u)\, du \\ &= \frac{1}{2} D_I^2 H_0(\mathcal{C})\eta^2 + H_1|_{\mathcal{C}}(\phi) - \int_0^\phi g_I|_{\mathcal{C}}(u)\, du,\end{aligned} \tag{99}$$

where the slow Hamiltonian $\mathcal{H}$ is defined in (62) and g_I is the I-component of the perturbation term g in Eq. (1). As seen from (98), for finite times the solutions on the manifold Π are approximated with an error of order $\mathcal{O}(\sqrt{\epsilon})$ by the level curves of the function $\mathcal{H}_g$. (We note that, in general, the flow generated by $\mathcal{H}_g$ is only locally Hamiltonian, i.e., it does not admit a single valued Hamiltonian on Π. We selected the Hamiltonian $\mathcal{H}_g$ in a way such that it generates the leading order Hamiltonian terms through the canonical symplectic form $d\phi \wedge dI$.)

Theorem 7.4. *Suppose that $m = 1$ and the conditions of Theorem 7.3 hold. Assume further that the curve $\{\phi = \phi_0\} \subset \Pi$ intersects transversely the unstable manifold of a hyperbolic fixed point $p_0 \in \Pi$ of the Hamiltonian $\mathcal{H}_g$. Let $(0, 0, \eta_0, \phi_0)$ be the coordinates of the point p_0 and assume that for any small enough $|z| > 0$ and $\epsilon > 0$, the point $(y^0, z, \eta_0, \phi_0 + N\Delta\phi) \in \mathcal{M}_\epsilon$ lies in the domain of attraction of an invariant set $\mathcal{S}_\epsilon \subset \Pi$.*

Then, for $\epsilon > 0$ sufficiently small, there exists $0 < \mu < \frac{1}{2}$ such that system (2) admits two N-pulse homoclinic orbits $x_\epsilon^\pm$ with basepoints $b_\epsilon^\pm = p_0 + \mathcal{O}(\epsilon^\mu) \in \Pi$ and with jump sequences $\chi^\pm(\phi_0)$, respectively. Both orbits are backward asymptotic to a hyperbolic fixed point $p_\epsilon = p_0 + \mathcal{O}(\sqrt{\epsilon}) \subset \Pi$ and forward asymptotic to the invariant set $\mathcal{S}_\epsilon$.

Proof. By Theorem 7.3 we immediately obtain the existence of a curve $\mathcal{B}_\epsilon \subset \Pi$ which contains basepoints for N-pulse homoclinic orbits of the type of $x_\epsilon^\pm$. From the proof of that theorem it is also clear that the curve $\mathcal{B}_\epsilon$ is C^1 $\mathcal{O}(\epsilon^\mu)$-close to the line $\{\phi = \phi_0\}$. As a result, it will intersect the unstable manifold of the fixed point p_ϵ, which perturbs from p_0 under the effect of dissipative and higher order Hamiltonian terms. Then, by the invariance properties of unstable fibers, this intersection point is a basepoint for an N-pulse homoclinic orbit that backward asymptotes to p_ϵ. Finally, the invariance properties of stable fibers imply that the N-pulse homoclinic orbit asymptotes to the attracting set $\mathcal{S}_\epsilon$ in forward time. □

In applications system (1) frequently depends on parameters. Varying these parameters on a codimension-one subset of the parameter space, it is possible to construct multi-pulse homoclinic orbits which have their basepoints precisely on an equilibrium point p_ϵ contained in the invariant manifold Π. If, in addition, the attracting set $\mathcal{S}_\epsilon$ assumed in the previous theorem is just the fixed point p_ϵ, then the multi-pulse homoclinic orbit obtained in this fashion is an orbit homoclinic to p_ϵ itself.

Theorem 7.5. *Suppose that $m = 1$, system (1) depends on a vector $\lambda \in \mathbb{R}^p$ of system parameters in a C^r fashion, and $V \subset \mathbb{R}^p$ is an open set. Assume further that*

(i) For any $\lambda \in V$ the Hamiltonian $\mathcal{H}_g$ has a nondegenerate equilibrium (i.e., no zero eigenvalues) $p_0(\lambda) = (\eta_0(\lambda), \phi_0(\lambda)) \in \Pi$.

(ii) *For some positive integer N and for some parameter value $\lambda_0 \in V$, $\phi_0(\lambda_0)$ satisfies the conditions of Theorem 7.3.*
(iii) $D_\lambda \Delta^N \mathcal{H}(\phi_0(\lambda), \lambda)|_{\lambda=\lambda_0} \neq 0$.
(iv) *For small enough $|z|$ and $\epsilon > 0$, the point $(y^0, z, \eta_0, \phi_0 + N\Delta\phi) \in \mathcal{M}_\epsilon$ lies in the domain of attraction of an asymptotically stable fixed point $p_\epsilon \subset \Pi$ of system (98) which perturbs from the fixed point p_0.*

Then there exists a codimension-one set $M^+ \subset \mathbb{R}^p \times \mathbb{R}$ near the point $(\lambda_0, 0)$ such that for every parameter value $(\lambda, \epsilon) \in M$, the system (2) admits an N-pulse homoclinic orbit x_ϵ^+ homoclinic to the point p_ϵ. The basepoint for this orbit is p_ϵ and the jump sequence of the orbit is given by $\chi^+(\phi_0(\lambda_0))$. There also exists another codimension one set $M^- \subset \mathbb{R}^p \times \mathbb{R}$ which yields similar homoclinic orbits with jump sequence $\chi^-(\phi_0(\lambda_0))$.

Proof. The main steps in the proof of this theorem are similar to those in the proof of Theorem 7.3. However, we now want to force the perturbed fixed point p_ϵ to be a solution of the equation

$$\Delta^N \mathcal{H}\,(p_\epsilon(\lambda); \lambda) + \delta_0 \mathcal{F}_N(p_N(p_\epsilon(\lambda)); \delta_0, \epsilon^\mu, \lambda) + \epsilon^\mu \mathcal{G}_N(p_N(p_\epsilon(\lambda)), \delta_0, \epsilon^\mu, \lambda) = 0$$

with $p_\epsilon(\lambda) = \left(\phi_0(\lambda) + \sqrt{\epsilon} P_1(\lambda, \epsilon), \eta_0(\lambda) + \sqrt{\epsilon} P_2(\lambda, \epsilon)\right)$. Using (iv) and the implicit function theorem, we see that this equation can again be solved in two steps to obtain a solution $\bar{\lambda}(\epsilon) = \lambda_0 + \mathcal{O}(\epsilon^\mu)$. □

We note that in the case of $n = 0$ the above theorem is identical to the one obtained in Haller and Wiggins [18] for the existence of Šilnikov-type orbits in two-degree-of-freedom systems. Another situation in which multi-pulse Šilnikov-type orbits may occur is when an equilibrium $p_\epsilon \in \Pi$ of the perturbed system is a saddle restricted to the manifold Π, but when viewed within the center manifold $\mathcal{M}_\epsilon$, it also admits n pairs of complex eigenvalues with negative real parts.

Theorem 7.6. *Suppose that $m = 1$ and system (1) depends on a parameter $\lambda \in \mathbb{R}$ in a C^r fashion. Let $V \in \mathbb{R}$ be an open set and assume that*

(i) *The Hamiltonian $\mathcal{H}_g$ has a nondegenerate equilibrium (i.e., no zero eigenvalues) $p_0(\lambda) = (\eta_0(\lambda), \phi_0(\lambda)) \in \Pi$. If $p_\epsilon(\lambda) \in \Pi$ is the corresponding equilibrium of the perturbed system (1), then the manifold $W^s(p_\epsilon(\lambda)) \cap \mathcal{M}_\epsilon$ is codimension one within the center manifold $\mathcal{M}_\epsilon$.*
(ii) *The "size" of $W^s(p_\epsilon(\lambda))$ is of order $\mathcal{O}(\epsilon^q)$ with $0 \leq q < 1$, i.e., it intersects a surface $|z| = K\epsilon^q$ transversely.*
(iii) *For some positive integer N and for all $\lambda \in V$, there exists a function $\phi_0(\lambda)$ which satisfies the conditions of Theorem 7.3.*
(iv) *The line $\{\phi = \phi_0(\lambda)\} \subset \Pi$ intersects transversely the unstable manifold of the fixed point $p_0 \in \Pi$ of the slow Hamiltonian.*
(v) *If $(0, 0, \eta_0(\lambda), \phi_0(\lambda))$ are the coordinates of this transverse intersection point, then the point $(0, 0, \eta_0, \phi_0(\lambda) + N\Delta\phi(\lambda))$ crosses the stable manifold of p_0 transversely as λ is varied through λ_0.*

Then there exists a codimension one set $M^+ \subset \mathbb{R}^2$ near the point $(\lambda_0, 0)$ such that for every parameter value $(\lambda, \epsilon) \in M$, the system (1) admits an N–pulse homoclinic orbit x_ϵ^+ to the point $p_\epsilon(\lambda)$. The basepoint for this orbit lies in $W^u(p_\epsilon) \cap \Pi$ and the jump sequence of the orbit is given by $\chi^+(\phi_0(\lambda_0))$. There also exists another codimension one set $M^- \subset \mathbb{R}^2$ which yields similar homoclinic orbits with jump sequence $\chi^-(\phi_0(\lambda_0))$.

Proof. Again, the main steps in the proof of this theorem coincide with those in the proof of Theorem 7.3. The new element is that we want to force the stable fiber, which is intersected by the N-pulse homoclinic orbit, to lie on the stable manifold of the perturbed fixed point p_ϵ. At the same time, we do not require the basepoint of the N-pulse orbit to coincide with p_ϵ as in the previous theorem, but rather we allow the basepoint to be any point in the set $W^u(p_\epsilon) \cap \Pi$.

As in the proof of Theorem 7.3, we first solve the equation

$$\Delta^N \mathcal{H}(p_\epsilon(\lambda); \lambda) + \delta_0 \mathcal{F}_N(p_N(p_\epsilon(\lambda)); \delta_0, \epsilon^\mu, \lambda) + \epsilon^\mu \mathcal{G}_N(p_N(p_\epsilon(\lambda)); \delta_0, \epsilon^\mu, \lambda) = 0$$

to obtain a solution $\bar{\phi}(\lambda, \epsilon) = \phi_0(\lambda) + \mathcal{O}(\epsilon^\mu)$. By assumption (iv) and by the C^1 dependence of $\bar{\phi}$ on ϵ^μ (cf. Theorem 7.3), the curve $\{\phi = \bar{\phi}(\lambda, \epsilon)\}$ intersects the unstable manifold of the fixed point $p_\epsilon(\lambda)$ transversely in a point

$$\bar{p}(\lambda, \epsilon) = (\eta_0(\lambda) + \mathcal{O}(\epsilon^\mu), \phi_0(\lambda) + \mathcal{O}(\epsilon^\mu)) \in \Pi.$$

We know (cf. Definition 7.1) that the N-pulse solution with basepoint $\bar{p}(\lambda, \epsilon)$ intersects a stable fiber $f^s(\hat{p}(\lambda, \epsilon))$ whose basepoint has the (y, z, η, ϕ) coordinates

$$\hat{p}(\lambda, \epsilon) = (0, \mathcal{O}(\epsilon), \eta_0(\lambda) + \mathcal{O}(\epsilon^\mu), \phi_0(\lambda) + \Delta\phi(\lambda) + \mathcal{O}(\epsilon^\mu)) \in \mathcal{M}_\epsilon. \tag{100}$$

Furthermore, by assumption (ii), in a vicinity of the manifold Π the stable manifold of p_ϵ can be written as a graph over either the (ϕ, z) or the (η, z) variables. Considering the former case (the latter can be dealt with in the same way), we obtain that near Π a compact subset of $W^s(p_\epsilon(\lambda))$ satisfies an equation of the form

$$\eta = m_1(\phi, \lambda) + z m_2(\phi, z, \lambda, \epsilon), \tag{101}$$

where m_j are of class C^r and $\eta = m_1(\phi, \lambda)$ is the local equation of the stable manifold of p_0 on the manifold Π.

Our goal is to find parameter values for which the stable fiber basepoint $\hat{p}(\lambda, \epsilon)$ is contained in the stable manifold of the fixed point $p_\epsilon(\lambda)$. From (100) we see that $\text{dist}(\hat{p}(\lambda, \epsilon), \Pi) = \mathcal{O}(\epsilon)$, and hence by assumption (ii) of the theorem, $\hat{p}(\lambda, \epsilon)$ lies in the domain where $W^s(p_\epsilon(\lambda))$ satisfies (101). Then formulas (100) and (101) give the equation

$$\begin{aligned} &\eta_0(\lambda) + \epsilon^\mu h_\eta(\lambda, \epsilon) - m_1\left(\phi_0(\lambda) + \epsilon^\mu h_\phi(\lambda, \epsilon), \lambda\right) \\ &-\epsilon h_z(\lambda, \epsilon) m_2\left(\phi_0(\lambda) + \epsilon^\mu h_\phi(\lambda, \epsilon), \epsilon h_z(\lambda, \epsilon), \lambda, \epsilon\right) = 0, \end{aligned} \tag{102}$$

where the functions h_η, h_ϕ, and h_z are differentiable in λ and ϵ^μ. Now by assumption (v), we know that

$$\eta_0(\lambda_0) - m_1(\phi_0(\lambda_0), \lambda_0) = 0, \quad D_\lambda \left[\eta_0(\lambda) - m_1(\phi_0(\lambda), \lambda)\right]_{\lambda=\lambda_0} \neq 0,$$

thus the implicit function theorem guarantees a solution $\bar{\lambda}(\epsilon) = \lambda_0 + \mathcal{O}(\epsilon^\mu)$ to Eq. (102). This completes the proof of the theorem. □

8. Geometry of the Unstable Manifold of Π

Using the methods of the proof of Theorem 7.3, we can follow any particular solution in the unstable manifold of the manifold Π on time scales of order $\mathcal{O}(\log 1/\sqrt{\epsilon})$, while the unstable manifold makes a finite number of "jumps". The following definition will be used to distinguish between different types of jumping orbits within the unstable manifold of Π.

Definition 8.1. *Let us consider a point $b_0 \in \mathcal{C}$ and let $j = \{j_i\}_{i=1}^N$ be a sequence of $+1$'s and -1's. An orbit x_ϵ of system (1) is called an N-*pulse orbit with basepoint b_0 and jump sequence j, *if for some $0 < \mu < \frac{1}{2}$ and for $\epsilon > 0$ sufficiently small,*

(i) x_ϵ intersects an unstable fiber $f^u(b_\epsilon)$ with basepoint $b_\epsilon = b_0 + \mathcal{O}(\epsilon^\mu) \in \Pi$.
(ii) Outside a small fixed neighborhood of the manifold $\mathcal{M}_\epsilon$, the orbit x_ϵ is order $\mathcal{O}(\sqrt{\epsilon})$ close to a chain of unperturbed heteroclinic solutions $x^i(t)$, $i = 1, \ldots, N$, such that

$$\lim_{t\to-\infty} x^1(t) = b_0, \qquad \lim_{t\to+\infty} x^{i-1}(t) = \lim_{t\to-\infty} x^i(t), \; i = 2, \ldots, N.$$

Furthermore, for $k = 1, \ldots, N$ and for all $t \in \mathbb{R}$ we have

$$x^k(t) \in \begin{cases} W_0^+(\mathcal{C}) \text{ if } j_k = +1, \\ W_0^-(\mathcal{C}) \text{ if } j_k = -1. \end{cases}$$

We have the following result for the existence of N-pulse orbits.

Theorem 8.1. *Suppose that for some positive integer N and for some $\phi_0 \in \mathbb{T}^m$ we have*

$$\Delta^k \mathcal{H}(\phi_0) \neq 0, \quad k = 1, \ldots, N-1.$$

Then, for $\epsilon > 0$ sufficiently small there exist constants $0 < \mu < \frac{1}{2}$ and $C_\eta > 0$, such that for any $0 \le |\eta_0| < C_\eta$, the system (2) admits two N-pulse orbits $x_\epsilon^\pm$ with basepoint $b_\epsilon \in \Pi$ such that $\phi_{b_\epsilon} = \phi_0 + \mathcal{O}(\epsilon^\mu)$ and $\eta_{b_\epsilon} = \eta_0$. The jump sequences of the orbits are given by $\chi^\pm(\phi_0)$, respectively.

Proof. Using the assumption of the theorem and the arguments from the proof of Theorem 7.3, we immediately conclude that for $\epsilon > 0$ small enough the inequalities

$$\Delta^k \mathcal{H}(\phi_0) + \delta_0 \mathcal{F}_k(p_k(b_\epsilon); \delta_0, \epsilon^\mu) + \epsilon^\mu \mathcal{G}_k(p_k(b_\epsilon); \delta_0, \epsilon^\mu) \neq 0$$

hold for $k = 1, \ldots, N-1$. As a result, the unstable manifold $W^u(\Pi)$ contains two N-pulse orbits with basepoint (η_0, ϕ_0). The jump sequences of these orbits can be found in exactly the same way as in the proof of Theorem 7.3. □

The above result can be used in examples to study the "disintegration" of the unstable manifold of Π. In particular, in the process of its jumping around Π, the open sets in the manifold $W^u(\Pi)$ depart from each other and follow different jump sequences. This results in observable irregular transient behavior near the broken homoclinic structure, even if there are no chaotic invariant sets created by the perturbation. We will use this fact when we apply our results to a discretization of the forced NLS equation.

9. An Alternative Formulation of the Results

It may happen that the unperturbed limit of system (1) admits an invariant which offers a more convenient base for perturbation methods than the Hamiltonian H_0. For this reason, we also present an easy modification of our results that uses some other integral of the unperturbed limit. This alternative formulation will prove very useful in our study of the discretized NLS equation in the next section.

We consider a modification of system (1) in the form

$$\dot{x} = \omega^{\sharp}(DH_0(x)) + \epsilon g(x), \tag{103}$$

and assume that for $\epsilon = 0$, there exists a C^{r+1} function $K_0 \colon \mathcal{P} \to \mathbb{R}$, which is independent of the Hamiltonian H_0 and Poisson commutes with H_0, i.e.,

$$\{H_0, K_0\} = \omega(\omega^{\sharp}(DH_0), \omega^{\sharp}(DK_0)) = 0. \tag{104}$$

This last condition implies that the flows generated by H_0 and K_0 through the symplectic form ω commute. We also assume that on the circle of equilibria $\mathcal{C}$,

$$DK_0|\mathcal{C} = 0. \tag{105}$$

Following the definition of the energy-difference functions in (84), we introduce the function

$$\boxed{\Delta^N \mathcal{K}(\phi) = -\sum_{i=1}^{N} \int_{-\infty}^{\infty} \langle DK_0, g\rangle \,|_{x^i(t)}\; dt.} \tag{106}$$

We also redefine the number σ in (85) as

$$\sigma = \text{sign}\,\left(DK_0 \cdot \mathbf{n}(p^+)\right), \tag{107}$$

as well as the sign sequences in Definition 7.2:

Definition 9.1. *For any value* $\phi_0 \in \mathbb{T}^m$, **the positive sign sequence** $\chi^+(\phi_0) = \{\chi_k^+(\phi_0)\}_{k=1}^N$ *is defined as*

$$\chi_1^+(\phi_0) = +1, \quad \chi_{k+1}^+(\phi_0) = \sigma\, sign\, \left(\Delta^k \mathcal{K}(\phi_0)\right) \chi_k^+(\phi_0), \qquad k = 1, \ldots, N-1.$$

The **negative sign sequence** $\chi^-(\phi_0) = \{\chi_k^-(\phi_0)\}_{k=1}^N$ *is defined as*

$$\chi^-(\phi_0) = -\chi^+(\phi_0).$$

We then have the following result.

Theorem 9.1. *The statements of Theorems 7.3-7.6 also hold if we replace the energy-difference function $\Delta^N \mathcal{H}$ with the function $\Delta^N \mathcal{K}$ defined in (106) and we use the definition of sign sequences given in Definition 9.1.*

Proof. Our estimates for the local dynamics near the manifold $\mathcal{M}_0$ in Sect. 4 as well as Lemma 6.1 make no use of the Hamiltonian H_1, hence they hold without change. Lemma 6.2 can also be proved using the function K_0 instead of H_0, noting that $H_1 \equiv 0$. Indeed, Eq. (105) ensures that K_0 has the same type of Taylor expansion near the resonant circle $\mathcal{C}$ as H_0 does. Furthermore, the change of K_0 along perturbed solutions during passages near the manifold $\mathcal{M}_\epsilon$ can be computed as (cf. (67))

$$\begin{aligned}\sum_{i=1}^{N-1} K_0(q_i) - K_0(p_i) &= \sum_{i=1}^{N-1} \int_0^{T_i^*} \dot{K}_0(x(t))\, dt \\ &= \sum_{i=1}^{N-1} \int_0^{T_i^*} \left[DK_0 \cdot \left(\omega^\sharp(H_0) + \epsilon g\right)\right]_{x(t)} dt \\ &= \sum_{i=1}^{N-1} \int_0^{T_i^*} \left[\{K_0, H_0\} + \langle DK_0, g\rangle\right]_{x(t)} dt \qquad (108) \\ &= \epsilon \sum_{i=1}^{N-1} \int_0^{T_i^*} \langle DK_0, g\,\rangle_{x(t)}\, dt,\end{aligned}$$

where we used (104). Moreover, this last integral can again be approximated (with error of order $\mathcal{O}(\delta_0)$) by an improper integral as in (71), because by (105), $|DK_0|$ decreases exponentially on the unperturbed solutions $x^i(t)$, hence the improper integral converges absolutely. Lemma 6.3 can also be stated in terms of DK_0 based on (105). The statement of Lemma 7.1 does not involve H_0 explicitly, so its proof remains the same. Based on all these lemmas, the main argument in the proof of Theorem 7.3 can be repeated using the invariant K_0 instead of H_0. In particular, one replaces the local coordinate w_2 in the representation of the global map $G_\epsilon(q_0)$ and the local map $L_\epsilon(p_1)$ with the value of K_0 at q_0 and p_1, respectively. This is possible because, in analogy with (82), we have

$$\langle DK_0(s_\epsilon(w_2, \zeta, \rho, \psi)), D_{w_2}s_\epsilon(w_2, \zeta, \rho, \psi)\rangle \neq 0,$$

since the vector DK_0 is perpendicular to perturbed trajectories up to an error of order $\mathcal{O}(\epsilon)$, and $D_{w_2}s_\epsilon$ encloses an angle of order $\mathcal{O}(1)$ with perturbed trajectories. As a result, we obtain the equation

$$\Delta^N \mathcal{K}(\phi_0) + \delta_0 \tilde{\mathcal{F}}_N(p_N(b_\epsilon); \delta_0, \epsilon^\mu) + \epsilon^\mu \tilde{\mathcal{G}}_N(p_N(b_\epsilon); \delta_0, \epsilon^\mu) = 0$$

for the basepoint b_ϵ of an N-pulse homoclinic orbit. This equation can again be solved for $\epsilon > 0$, if we apply the implicit function theorem using the extension L_0 of the map L_ϵ. Adapting the definition of sign sequences from Definition 9.1, the jump sequences of N-pulse orbits can be constructed in exactly the same way as in the proof of Theorem 7.3. Finally, we can repeat the proofs of Theorems 7.4-7.4 without any change using the function $\Delta^N \mathcal{K}$ instead of $\Delta^N \mathcal{H}$. □

10. Jumping Homoclinic Orbits in a Discretization of the Perturbed NLS Equation

Let us consider the periodically forced and damped, focusing nonlinear Schrödinger equation

$$iu_t - u_{xx} - 2\,|u|^2\,u = i\epsilon(\Gamma e^{i2\Omega^2 t} - \alpha u + \beta u_{xx}), \tag{109}$$

with constants $\Omega, \Gamma, \alpha, \beta > 0$, and with the small parameter $\epsilon > 0$. We assign even, periodic boundary conditions of the form

$$u(x,0) = u(-x,0), \qquad u(x+1,t) = u(x,t).$$

Introducing the change of variable $u \to ue^{-i2\Omega^2 t}$, we can rewrite (109) as

$$iu_t - u_{xx} - 2\left[|u|^2 - \Omega^2\right]u = i\epsilon(\Gamma - \alpha u + \beta u_{xx}). \tag{110}$$

For $\beta = 0$, this equation agrees with the form of the perturbed NLS that was studied by Bishop *et al.* [5] as a small amplitude approximation to the parametrically forced sine-Gordon Eq. (see Sect. 1 for further references). For $\beta > 0$ we obtain a form of the perturbed NLS whose modal truncation and discretization was studied in the references listed in Sect. 1.1.

The $\epsilon = 0$ limit of Eq. (110) admits a discretization which was pointed out to be integrable for arbitrary mesh size by Ablowitz and Ladik [1]. Applying this particular discretization with mesh size $h > 0$ to the $\epsilon > 0$ case yields the system of ordinary differential equations

$$\begin{aligned} \dot{u}_k = &-i\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} - i\,|u_k|^2\,\big(u_{k-1} + u_{k+1}\big) + 2i\Omega^2 u_k \\ &+\epsilon\left(\Gamma - \alpha u_k + \beta\left(\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2}\right)\right), \end{aligned} \tag{111}$$

where $u_k(t) = u(x_k, t)$, $k = 0, \ldots, K-1$, $x_0 = -1$, $x_k = x_1 + kh$. The periodic boundary conditions for the PDE imply that $u_K(t) \equiv u_0(t)$. For $\epsilon = 0$, system (111) (together with the conjugate equations for $\bar{u}_k$), is Hamiltonian with Hamiltonian

$$H_0 = \frac{1}{h^2}\sum_{k=1}^{K-1}\left[\bar{u}_k(u_{k+1} + u_{k-1}) - \frac{2}{h^2}(1 + \Omega^2 h^2)\log(1 + h^2|u_k|^2)\right], \tag{112}$$

and with the symplectic form

$$\omega = \sum_{k=0}^{K-1} \frac{i}{2(1 + h^2|u_k|^2)}\mathrm{Im}(d\bar{u}_k \wedge du_k).$$

The discretization (111) gives a tool for approximating solutions of the partial differential Eq. (110), and also offers a finite dimensional model for the phase space structure of the perturbed NLS. In particular, for $\epsilon = 0$, (111) is integrable (see Ablowitz and Ladik [1] and Li and McLaughlin [31]). This is a special feature of this discretization which distinguishes it from the standard finite difference discretization of the NLS. (The usual finite difference scheme would have the same linear part but a nonlinear term of the form $u_k|u_k|^2$, which would only ensure integrability for $K = 2$.) System (111) also admits a two-dimensional invariant plane Π given by

$$u_1 = u_2, \;\; u_2 = u_3, \;\; \ldots \;\; u_{K-1} = u_K. \tag{113}$$

This plane is the set of solutions with no spatial dependence, and it is easily seen to remain invariant for $\epsilon > 0$. Restricting the dynamics to Π as in (113), one obtains the equation

$$\dot{u}_K = 2i\left[\Omega^2 - |u_K|^2\right] u_K + \epsilon(\Gamma - \alpha u_K). \tag{114}$$

This shows that for $\epsilon = 0$, Π contains a circle of equilibria $\mathcal{C}$ which is given by $|u_K| = \omega$. The circle $\mathcal{C}$ is surrounded by periodic solutions in the plane Π. Introducing the action-angle variables $(I, \phi) \in \mathbb{R} \times S^1$ by letting $u_K = Ie^{i\phi}$, we can rewrite Eq. (114) in the form

$$\begin{aligned} \dot{I} &= \epsilon(\Gamma \cos\phi - \alpha I), \\ \dot{\phi} &= 2(\Omega^2 - I^2) - \epsilon\frac{\Gamma}{I}\sin\phi. \end{aligned} \tag{115}$$

For $\epsilon = 0$, this system is a one-degree-of-freedom Hamiltonian system with Hamiltonian

$$H_\Pi \equiv H_0|\Pi = \frac{2K}{h^2}\left[I^2 - \frac{1}{h^2}(1 + \Omega^2 h^2)\log(1 + h^2 I^2)\right],$$

and with the symplectic form

$$\omega_\Pi \equiv \omega|\Pi = \frac{-KI}{2(1 + h^2 I^2)} d\phi \wedge dI,$$

which is clearly nondegenerate. Linearizing (111) about any point of the circle $\mathcal{C}$, one finds that for

$$\begin{aligned} 3\tan\frac{\pi}{K} &< \Omega < \infty, && \text{if } K = 3, \\ K\tan\frac{\pi}{K} &< \Omega < K\tan\frac{2\pi}{K}, && \text{if } K > 3, \end{aligned} \tag{116}$$

off the plane Π, the linearized system possesses one positive, one negative, and $K - 2$ pairs of pure imaginary eigenvalues (see Li and McLaughlin [31]). Furthermore, the circle $\mathcal{C}$ admits a codimension two center manifold $\mathcal{M}_0$ which contains the plane Π. For $\epsilon = 0$, Li [29] showed the existence of an $n-2$ parameter family of orbits homoclinic to the center manifold $\mathcal{M}_0$, which implies the existence of a codimension one homoclinic manifold $W^u(\mathcal{M}_0) \equiv W^s(\mathcal{M}_0)$. A three dimensional submanifold of this homoclinic structure carries motions that are doubly asymptotic to the plane Π itself, hence we obtain that $W^u(\mathcal{C}) \equiv W^s(\mathcal{C}) = W_0^+ \cup W_0^-$. Here $W_0^\pm$ denote the two connected components of the manifold homoclinic to Π. This manifold is filled with heteroclinic orbits connecting points on the circle $\mathcal{C}$. As shown in Li [29], the phase shift along all these heteroclinic connections is given by

$$\Delta\phi = -4\tan^{-1}\frac{\sqrt{[1 + \frac{\Omega^2}{K^2}]\cos\frac{\pi}{K} - 1}}{\sqrt{1 + \frac{\Omega^2}{K^2}}\sin\frac{\pi}{K}}. \tag{117}$$

If we pass to the real coordinates $(\phi_k, I_k) \in \mathbb{R} \times S^1$ by letting $u_k = I_k e^{i\phi_k}$, then the discretized NLS equation is of the form (1) with

$$\begin{aligned} H_0 &= \frac{2}{h^2}\sum_{k=0}^{K-1}\left[I_k e^{-i\phi_k}(I_{k+1}e^{i\phi_{k+1}} + I_{k-1}e^{i\phi_{k-1}}) - \frac{2}{h^2}(1 + \Omega^2 h^2)\log(1 + h^2 I_k^2)\right], \\ H_1 &\equiv 0, \end{aligned} \tag{118}$$

$$\omega = \sum_{k=0}^{K-1} \frac{-I_k}{2(1+h^2 I_k^2)} d\phi_k \wedge dI_k, \quad g = G(I, \phi; \alpha, \beta, \Gamma).$$

Furthermore, based on the above description of system (111), the resulting real system of equations satisfies assumptions (H1)-(H7) of Sect. 2 with $m = 1$ and $n = 2(K-2)$. As a result, the theory we have developed in this paper can be used to investigate the existence of multi-pulse homoclinic orbits for the discretized NLS system (111). First, we will study the equations setting $\beta = 0$ which was the case in the study of Bishop *et al.* [5, 6]. Later, we will consider the case $\beta > 0$, which was studied first in Li and McLaughlin [31].

10.1. The $\beta = 0$ limit. As described in Li and McLaughlin [31], the unperturbed integrable system admits an invariant denoted $\tilde{F}_1$ such that

$$\tilde{F}_1 | \mathcal{C} = 0. \tag{119}$$

The function F_1 is defined as a Floquet discriminant computed for a set of fundamental solutions to a discretized Lax pair for system (111). For brevity, we do not introduce here all the notation and terminology for the exact definition of F_1, but refer the reader to Li and McLaughlin [29]. All we need in our analysis is the existence of F_1 and the results of some involved calculations performed in [31]. In particular, using an implicit derivation, Li and McLaughlin [29] computed a Melnikov integral to study the existence of (single-pulse) homoclinic orbits for system (111). They obtain that, for $\beta = 0$, the Melnikov integral computed on unperturbed orbits homoclinic to the circle $\mathcal{C}$ can be written as

$$\hat{M}_{F_1}(\phi) = \int_{-\infty}^{\infty} \langle D\tilde{F}_1, g\rangle|_{x^h(t)}\, dt = \Gamma \left[\hat{M}_\Gamma \cos\left(\phi + \frac{\Delta\phi}{2}\right) - \chi_\alpha M_\alpha \right]. \tag{120}$$

Here the nonzero constants M_Γ and M_α depend only on the number Ω and the mesh size K of the discretization, $\chi_\alpha = \alpha/\Gamma$, and the phase shift $\Delta\phi$ is defined in (117). The heteroclinic solution $x^h(t)$ has the property that for its $\phi_k(t)$ component

$$\lim_{t\to-\infty} \phi_k(t) = \phi, \quad k = 0, \ldots, K-1$$

holds, where $\phi \in S^1$ is the argument of $\hat{M}_{F_1}$.

By (118), the real system corresponding to (111) can in fact be written in the form (103). This fact together with (119) implies that the alternative formulation of our main results in Sect. 8 applies to the discretized NLS system. To find multi-pulse homoclinic orbits, we have to study the zeros of the function $\Delta^N \mathcal{K}$ defined in (106). Setting $K_0 = \tilde{F}_1$ and using (120), we obtain that

$$\begin{aligned} \Delta^N \mathcal{K}(\phi) &= -\sum_{i=1}^{N} \int_{-\infty}^{\infty} \left\langle D\tilde{F}_1, g \right\rangle |_{x^i(t)}\ dt \\ &= -\Gamma \left[\hat{M}_\Gamma \sum_{k=0}^{N-1} \cos\left(\phi + \frac{2k+1}{2}\Delta\phi\right) - N\chi_\alpha M_\alpha \right]. \end{aligned} \tag{121}$$

Using the relation

$$\sum_{k=0}^{N-1}\cos\left(\phi+\frac{2k+1}{2}\Delta\phi\right)=\mathrm{Re}\sum_{k=0}^{N-1}e^{i[\phi+(2k+1)\Delta\phi/2]}=\mathrm{Re}\,\frac{e^{i(\phi+\Delta\phi/2)}\left(e^{iN\Delta\phi}-1\right)}{e^{i\Delta\phi}-1}$$
$$=\frac{\sin\frac{N\Delta\phi}{2}}{\sin\frac{\Delta\phi}{2}}\cos\left(\phi+\frac{N\Delta\phi}{2}\right),$$

we obtain that

$$\Delta^N\mathcal{K}(\phi)=-\Gamma\left[\frac{\hat{M}_\Gamma\sin\frac{N\Delta\phi}{2}}{\sin\frac{\Delta\phi}{2}}\cos\left(\phi+\frac{N\Delta\phi}{2}\right)-N\chi_\alpha M_\alpha\right]. \tag{122}$$

If

$$\Delta\phi\neq\frac{2j\pi}{N},\quad j\in\mathbb{Z}, \tag{123}$$

and

$$N\left|\chi_\alpha M_\alpha\sin\frac{\Delta\phi}{2}\right|\leq\left|\hat{M}_\Gamma\sin\frac{N\Delta\phi}{2}\right|, \tag{124}$$

then $\Delta^N\mathcal{K}$ admits two transverse zeros given by

$$\begin{aligned}\phi_1^N&=\frac{\pi}{2}-\frac{N\Delta\phi}{2}-\cos^{-1}\frac{N\chi_\alpha M_\alpha\sin\frac{\Delta\phi}{2}}{\hat{M}_\Gamma\sin\frac{N\Delta\phi}{2}},\\ \phi_2^N&=\frac{3\pi}{2}-\frac{N\Delta\phi}{2}-\cos^{-1}\frac{N\chi_\alpha M_\alpha\sin\frac{\Delta\phi}{2}}{\hat{M}_\Gamma\sin\frac{N\Delta\phi}{2}}.\end{aligned} \tag{125}$$

Using these zeros, we can obtain the following result.

Theorem 10.1. *Consider any integer $N\geq 1$ and suppose that*

$$\tan\left(-\frac{j\pi}{2N}\right)\sin\frac{\pi}{2}\neq\sqrt{\frac{[1+\frac{\Omega^2}{K^2}]\cos\frac{\pi}{K}-1}{1+\frac{\Omega^2}{K^2}}},\quad j\in\mathbb{Z}. \tag{126}$$

Assume further that conditions (123) and (124) hold.
Then, for $\epsilon,\alpha>0$ sufficiently small,

(i) The discretized NLS system (111) admits four, 1-parameter families of N-pulse homoclinic orbits, which are backward asymptotic to the invariant plane Π and forward asymptotic to a codimension two invariant manifold $\mathcal{M}_\epsilon$, which contains Π. The coordinates of the basepoints of the N-pulse homoclinic orbits are of the form

$$u_1^{l,\pm}=u_2^{l,\pm}=\ldots=u_K^{l,\pm}=(\Omega+\mathcal{O}(\sqrt{\epsilon})e^{i(\phi_l^N+\mathcal{O}(\sqrt{\epsilon}))},\quad l=1,2.$$

(ii) The jump sequences of the orbit families satisfy

$$j^{l,+}=-j^{l,-},\quad l=1,2. \tag{127}$$

Furthermore, every time the jump sequence $j^{l,\pm}$ changes sign, the jump sequence $j^{m,\pm}$ with $l\neq m$ will not change sign, and vice versa.

(iii) *The unstable manifold of the invariant plane Π contains $4N$ families of N-pulse orbits (see Definition 8.1) such that all these families have different jump sequences.*

Proof. We first note that formula (126) ensures that the zeros of the function $\Delta^N\mathcal{K}$ are transverse (see (117) and (123)). Next we observe that for $\alpha = 0$, condition (123) guarantees that $\phi_l^N \neq \phi_m^k$ with $l, m \in \{1, 2\}$ and $k = 1, \ldots, N-1$. This implies that for any fixed N, $\Delta^k\mathcal{K}(\phi_l^N) \neq 0$, $k = 1, \ldots, N-1$. This property is clearly preserved for $\alpha > 0$ sufficiently small, hence Theorem 7.3 implies statements (i). By Theorem 7.3, the two families corresponding to the zero ϕ_l^N have opposite jump sequences, which is stated in Eq. (127).

To prove the second statement in (ii) about sign changes in the jump sequences, we note that for $\alpha = 0$ and for any $k \in \mathbb{Z}$, we have

$$\operatorname{sign} \Delta^k\mathcal{K}(\phi_1^N) = -\operatorname{sign} \Delta^k\mathcal{K}(\phi_2^N),$$

since the minimal period of $\Delta^k\mathcal{K}$ is 2π and the difference between the zeros ϕ_1^N and ϕ_2^N is exactly equal to π. But for sufficiently small $\alpha > 0$, this last equation together with the definition of the sign sequence $\chi^{\pm}(\phi_l^N)$, and the fact that $j^{l,\pm} = \chi^{\pm}(\phi_l^N)$, implies the second statement in (ii).

Statement (iii) follows directly from Theorem 7.6 for $\alpha > 0$ small, because the $2N$ disjoint lines $\{\phi = \phi_l^k\}_{k=1,\ldots,N,l=1,2}$ divide the plane Π into $2N$ sectors, so that one of the functions $\Delta^k\mathcal{K}$ always changes sign at the boundary of these sectors. □

According to statement (ii) of the above theorem, *if there are homoclinic orbits that, for at least some of their pulses, stay near one particular component of the unperturbed homoclinic structure $W_0(\mathcal{M}_0)$, then there are other multi-pulse orbits that keep switching between different components of $W_0(\mathcal{M}_0)$.*

Theorem 10.1 does not identify the exact asymptotics of the multi-pulse solutions. The asymptotic behavior of these orbits could be identified using Theorems 7.4 or 7.5, and a likely candidate for the attracting set $\mathcal{S}_\epsilon$ is a sink created by the perturbation in the plane Π. The role of the hyperbolic fixed point $p_0 \in \Pi$ is then played by a saddle point on Π. (The existence of these fixed points is easy to verify from Eq. (114).) However, the identification of the domain of attraction of $\mathcal{S}_\epsilon$ leads to extensive calculations in this example. Nevertheless, the results of Li and McLaughlin [31] indicate that the conditions of Theorem 7.4 are indeed satisfied, which suggests the existence of the same type of jumping heteroclinic orbits between the two equilibria as those described in Haller and Wiggins [19].

10.2. The case of $\beta > 0$. For the case of $\beta > 0$, the calculations of the previous subsection leading to the expressions (125) can be repeated. Using the formulas of Li and McLaughlin [31], one obtains in the same fashion that

$$\Delta^N\mathcal{K}(\phi) = -\Gamma\left[\frac{\hat{M}_\Gamma \sin\frac{N\Delta\phi}{2}}{\sin\frac{\Delta\phi}{2}}\cos\left(\phi + \frac{N\Delta\phi}{2}\right) - N\left(\chi_\alpha M_\alpha - \chi_\beta M_\beta\right)\right], \quad (128)$$

where $\chi_\beta = \beta/\Gamma$ and M_β is a nonzero constant that depends on the parameter Ω and the mesh size K only. It is easy to see that the roots of this equation are smooth in χ_β, therefore the results listed in Theorem 10.1 remain valid for sufficiently small $\beta > 0$ (cf. the proof of that theorem). Instead of repeating these results, we will use Theorem 7.6 to construct multi-pulse homoclinic orbits to a fixed point $p_\epsilon \in \Pi$. These orbits will be the multi-pulse analogs of the single-pulse homoclinic orbits constructed by Li and McLaughlin in [31].

Theorem 10.2. *Let N be an arbitrary but fixed positive integer, and let $\Omega > 0$ be a constant such that conditions (116) and (123) are satisfied. Let the mesh size $K \geq 3$ be an integer for which the codimension one surface*

$$M_0 = \left\{ (\alpha, \beta, \Gamma, \epsilon) \middle| \, \beta = \frac{\alpha}{M_\beta} \left(M_\alpha - \Omega \hat{M}_\Gamma \frac{\Delta\phi \sin \frac{N\Delta\phi}{2}}{2 \sin^2 \frac{\Delta\phi}{2}} \right), \right. \\ \left. |\chi_\alpha M_\alpha - \chi_\beta M_\beta| < \frac{\left| \hat{M}_\Gamma \sin \frac{N\Delta\phi}{2} \right|}{N \left| \sin \frac{\Delta\phi}{2} \right|}, \; \epsilon < \epsilon_0 \right\} \tag{129}$$

of the $(\alpha, \beta, \Gamma, \epsilon)$ parameter space is nonempty.

Then there exists $\epsilon_0 > 0$ and two codimension one surfaces $M_\epsilon^\pm \in \mathbb{R}^{4+}$ with the following properties:

(i) *$M_\epsilon^\pm$ is $\mathcal{O}(\epsilon^q)$ C^0-close to the surface M_0 in the $(\alpha, \beta, \Gamma, \epsilon)$ parameter space.*
(ii) *For every $(\alpha, \beta, \Gamma, \epsilon) \in M_\epsilon^\pm$, system (111 admits an N-pulse homoclinic orbit which is doubly asymptotic to a fixed point $p_\epsilon \in \Pi$. The coordinates of p_ϵ are given by $(\eta_{p_\epsilon}, \phi_{p_\epsilon}) = \left(0, \cos^{-1}(\chi_\alpha \Omega)\right) + \mathcal{O}(\sqrt{\epsilon})$.*
(iii) *The basepoint of the N-pulse homoclinic orbit lies on the unstable manifold of p_ϵ, and the jump sequence of the orbit starts with ± 1.*

Proof. We only have to verify conditions (i)-(v) of Theorem 7.6, from which the statements of the present theorem follow directly. We first recall that compact segments of the orbits on the invariant plane Π can be approximated by the level curves of the Hamiltonian $\mathcal{H}_g$ defined in (99), which in this case takes the form

$$\mathcal{H}_g(\eta, \phi) = -2\Omega^2 \eta^2 - \Gamma \sin \phi + \alpha \Omega \phi, \tag{130}$$

as one obtains by Taylor expanding the right hand side of (115). The level curves of this Hamiltonian are shown in Fig. 5. Note that $p_0(\chi_\alpha) = \left(0, \cos^{-1}(\chi_\alpha \Omega)\right)$ is a saddle point

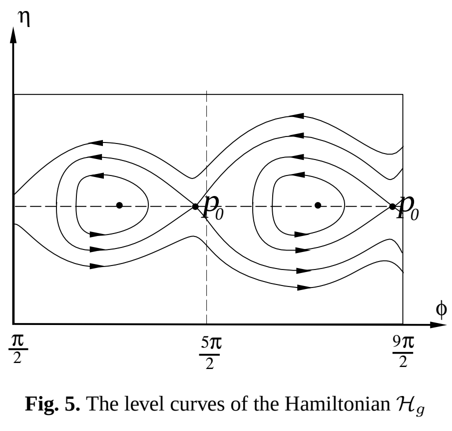

Fig. 5. The level curves of the Hamiltonian $\mathcal{H}_g$

with a homoclinic loop. As shown in Li and McLaughlin [31], for $\epsilon > 0$ the equilibrium $p_\epsilon \in \Pi$ perturbing from p_0 admits n pairs of complex eigenvalues with negative real parts, thus we obtain that $W^s\,(p_\epsilon(\chi_\alpha)) \cap \mathcal{M}_\epsilon$ is a codimension one surface within the manifold $\mathcal{M}_\epsilon$. Consequently, assumption (i) of Theorem 7.6 is satisfied. Condition (ii) of Theorem 7.6 is established in Sect. 6 of Li and McLaughlin [31] with reference to Li *et al.* [33]. Condition (iii) of Theorem 7.6 is satisfied if

$$|\chi_\alpha M_\alpha - \chi_\beta M_\beta| < \frac{\left|\hat{M}_\Gamma \sin \frac{N\Delta\phi}{2}\right|}{N\left|\sin \frac{\Delta\phi}{2}\right|}, \tag{131}$$

in which case the function $\Delta^N \mathcal{K}(\phi)$ defined in (128) has a zero ϕ_0. Note that (131) is satisfied for parameter values taken from the set M_0 , and the transversality of these zeros is guaranteed by condition (123). Therefore, assumption (iii) of Theorem 7.6 is satisfied. The validity of assumption (iv) can be seen from the phase portrait in Fig. 5.

To prove the theorem, it remains to verify condition (v) of Theorem 7.6. This can be done by adapting the distance measurement used in McLaughlin *et al.* [34] and Li and McLaughlin [31] as follows. The point $\hat{p} = (0, 0, \eta_0(\chi_\alpha), \phi_0(\chi_\alpha) + N\Delta\phi)$ lies on the stable manifold of p_0 if the value of Hamiltonian $\mathcal{H}_g$ at $\hat{p}$ is the same as at some other point of the homoclinic loop attached to p_0. In particular, it suffices to require that

$$\mathcal{H}_g(\eta_0(\chi_\alpha), \phi_0(\chi_\alpha)) = \mathcal{H}_g(\eta_0(\chi_\alpha), \phi_0(\chi_\alpha) + N\Delta\phi).$$

From (130) we obtain that this last equation can be written in the form

$$2\Gamma \cos\left(\phi_0(\chi_\alpha) + \frac{N\Delta\phi}{2}\right) \sin\frac{\Delta\phi}{2} - \alpha\Omega N\Delta\phi = 0. \tag{132}$$

Using the expression of $\Delta^N \mathcal{K}(\phi)$ and the fact that $\phi_0(\chi_\alpha)$ is a zero of $\Delta^N \mathcal{K}(\phi)$, we can rewrite (132) as

$$\chi_\beta - \frac{\chi_\alpha}{M_\beta}\left(M_\alpha - \Omega\hat{M}_\Gamma \frac{\Delta\phi \sin\frac{N\Delta\phi}{2}}{2\sin^2\frac{\Delta\phi}{2}}\right) = 0.$$

The transverse crossing of the unstable manifold of p_0 by the point $\hat{p}$ is equivalent to the left hand side of this equation admitting a nonzero derivative with respect to, e.g., the parameter χ_α at a solution $\chi_\alpha(\beta, \Gamma, \Omega, N)$. Since the equation is linear in χ_α, this transversality condition clearly holds, thus condition (v) of Theorem 7.6 is satisfied. This concludes the proof of the theorem. □

We remark that Li and McLaughlin [31] showed that the set M_0 defined in the statement of the above theorem is nonempty for $K > 7$ and for $N = 1$ (i.e., for single-pulse homoclinic orbits). We also note that the multi-pulse homoclinic orbits obtained from the theorem have the same asymptotic behavior as the single-pulse homoclinic orbits, hence the construction of chaotic invariant sets in their vicinities can be directly adapted from Li and Wiggins [32].

11. Conclusions

In this paper we gave a general criterion for the existence of nontrivial homoclinic orbits in a large class of near-integrable, multi-dimensional systems that usually arise as modal truncations or discretizations of partial differential equations. The homoclinic orbits we constructed make repeated departures from, and returns to, a codimension two invariant manifold which carries solutions with a slow and a fast time scale. The shape of the pulses (i.e., excursions of the homoclinic orbits) can be described by a sequence of +1s and -1s which we compute explicitly. Our results generalize the *energy-phase method* in Haller [16] and Haller and Wiggins [19] to arbitrarily high (but finite) dimensional systems.

We remark that if the perturbation in Eq. (1) is purely Hamiltonian (i.e., $g \equiv 0$), then the multi-pulse orbits generically undergo a sequence of *universal bifurcations* as the parameters of the system are varied. Such a bifurcation has been first described in an example in Haller [15] and then were shown to be generic near double resonances of near-integrable Hamiltonian systems in Haller [15]. Since for purely Hamiltonian perturbations, the energy-difference function $\Delta^N \mathcal{H}$ obtained in this paper is the same as in [15], the same universality holds for the bifurcations of multi-pulse orbits in system (1).

As an application of our results, we showed that the discretized, perturbed NLS equation admits multi-pulse solutions homoclinic to its center manifold. In fact, the pulse number of these orbits can be arbitrarily high if the dissipative and forcing terms are small enough. Statement (ii) of Theorem 10.1 also shows that N-pulse orbits with quite different shapes will coexist. Furthermore, statement (iii) describes how the unstable manifold of the plane Π disintegrates through multi-pulse jumping into components which display completely different jumping behaviors. Since the multi-pulse orbits spend a time of order $\mathcal{O}\left(\log 1/\sqrt{\epsilon}\right)$ (as opposed to $\mathcal{O}(1/\sqrt{\epsilon})$ as in Kaper and Kovačič [26]) in the neighborhood of the manifold $\mathcal{M}_\epsilon$, they have observable open neighborhoods in which solutions exhibit the same type of jumping behavior for finite times. Given the close coexistence of multi-pulse orbit families with different jump sequences, one expects to see a transient type of chaotic dynamics in numerical simulations. This agrees well with the irregular jumping behavior observed by Bishop *et al.* [5, 6] for $\beta = 0$.

Finally, we also considered the discretized NLS equation with a mode-dependent damping term ($\beta \neq 0$). Making use of the calculations of Li and McLaughlin [31], we showed the existence of multi-pulse Silnikov-type homoclinic orbits for a codimension one set of parameter values. This provides a significant extension of the set of parameter values for which the discretized NLS equation admits chaotic invariant sets in its phase space.

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